
Collegium Universitatis
in Academiâ Oxon.

R, in *Pater-*
Church-yard;
XXIII.

AN ANALYTICK TREATISE OF Conick Sections,

And their USE for Resolving of EQUATIONS in Determinate and Indeterminate PROBLEMS.

BEING

The Posthumous Work of the Marquis De L'HOSPITAL, Honorary Fellow of the Academy Royal of Sciences.

Made English *by* E. STONE.

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THE

TRANSLATOR

TO THE

READER.

CONICK SECTIONS *being Curves, by the Knowledge whereof such vast Discoveries and Improvements have been made both in* Astronomy, Geometry, Algebra, *and* Natural Philosophy; *and our Illustrious and Learned Author finding that most of their useful Properties and Problems might be demonstrated shorter and easier than in any publick* Treatise *of them; as also that there was wanting a general and easy way of drawing the* Conick Sections, *being the Loci of Equations of two Dimensions, with the manner of constructing Equations and solving Geometrical Problems by the Loci, did therefore write the following* Treatise *of* Conick Sections, *and*

their Use in Geometry *and* Algebra; *but died about the Time he intended to publish it, and left the Manuscript without a Preface; he thinking, I suppose, that the Title thereof, together with his great Character in Things of this Nature, would well enough supply that Deficiency.*

This Work is divided into Ten Books, whereof the first Three treat separately of the Parabola, Ellipsis, *and* Hyperbola. *In these Books are laid down the fundamental Properties and Problems of those Curves demonstrated algebraically, after a very short and easy manner, from their most simple and natural Descriptions upon a Plane. They being particularly adapted to the Capacities of those who are acquainted with but a few of the Propositions of the first six Books of* Euclid, *and withal have a small Knowledge in the Rudiments of* Algebra.

And because those Properties attending the Ellipsis, do also in some wise appertain to the Hyperbola, and opposite Sections, and even sometimes to the Parabola; therefore our Noble Author, in the Fourth Book of the Three Conick Sections, *lays down his Propositions more general than in the Three former Books; for the Propositions here do contain that Property of one, both, or all three of the Curves that was separately shewn in those Three Books.* Here *are also several other Properties and Problems not in the former Three Books: All demonstrated after a short and easy way.*

The Fifth Book contains the Comparison of the Conick Sections, and their Segments with each other, *in several Propositions and Corollaries, with a short Specimen of the Method of drawing Tangents to Curves, demonstrated after as short and easy manner as the Nature of the thing*

thing will admit, from three or four fundamental Lemmata.

Altho' it be harder, especially for one who has not read the 11*th Book of* Euclid, *to comprehend the Demonstrations of the fundamental Properties of the* Conick Sections *from the Cone, than from their Description upon a Plane by means of the noted Properties of the Foci, on account of the Consideration of Planes; yet our Noble Author, thinking that some People would rather like this Way, as being easier to those who are unacquainted with* Algebra, *and that his Work would be imperfect without it, has given us a Tract in his Sixth Book,* Of the Conick Sections consider'd in the Solid. *But herein has proceeded after a different Way from other Authors: For he considers the Ellipsis as the Section of a Cylinder, and easily demonstrates the Properties of its Diameters from the Cylinder; and afterwards, by supposing Cones to have Elliptick Bases instead of Circular ones, he easily proves the same Properties of the Parabola and Hyperbola.*

The Seventh Book is of Geometrick Loci, *which are treated of more easy and general than elsewhere, especially the Loci, which are* Conick Sections.

The Eighth Book is of Indeterminate Problems, *and shews the Use of the Seventh Book of Geometrick Loci in resolving them.*

The Ninth Book is of the Construction of Equations. *Wherein by the Intersection of two Loci, you are taught how to construct any Equation of what Dimension soever.*

Lastly, the Tenth Book of Determinate Problems *is a farther Use of the Loci in solving them. Herein are some*

some pleasant and new Theorems touching the Inscription of regular Polygons in a Circle, and a Demonstration of Sir Isaac Newton*'s* Theorem for finding the Unciæ.

Now because we have nothing in English *on this Subject but what is lame and imperfect, and there are several Persons who would be desirous of reading this Treatise in* English, *I therefore present them with it; and here shall add to the same the following Proposition, with an easy Demonstration, which is of great Use in the Doctrine of Centripetal Forces, and consequently in Astronomy; and for what Reason, I know not, omitted by our Illustrious Author.*

ANY Parallelogram (vide *Plate* 31. *Fig.* B.C.) *described about an Ellipsis, or between the Conjugate Hyperbola's, so that the four Points of Contact may be join'd by two Diameters* G H, I F *only, which therefore will be Conjugates, is equal to the Parallelogram describ'd about the two Axes* A a, B b; *and consequently all such Parallelograms are equal to one another.*

From F, the Extremity of one Diameter, draw the Line FD parallel to the other Diameter GH, (continu'd out in the opposite Hyperbola's) meeting the Axis (produced in the Ellipsis) in the Point T, and from G the Extremity of the Diameter GH draw the Line GD parallel to the Diameter IF meeting DF in D. And from the Point F let fall the Perpendicular FP to the Axis Aa. Then GD, DF will * touch the Ellipsis, and the Hyperbola's bG, aF in G, F the Extremities of the Conjugate Diameters, and so the Parallelogram $CGDF$ will be $\frac{1}{4}$ of that describ'd about the Ellipsis, or between the Conjugate Hyperbola's, having the Condition mention'd in the Theorem. Therefore, if CE be drawn perpendicular to DF (produced in the Conjugate Hyperbola's) we are to prove that CG $(= DF) \times CE$ is $= Cb \times Ca = \frac{1}{4}$ of the Rectangle under the two Axes.

* *Art.* 56, 134. *and Def.* 13. *Book* III.

Call Ca, t; Cb, c; and Cp, x; then $\overline{Ca}^2$ $(tt$ *$)$: $AP \times Pa$ $(tt - xx$ in Ellipsis, or $xx - tt$ in Hyperbola$)$: : $\overline{Cb}^2$ (cc) : $\overline{FP}^2 = cc - \frac{ccxx}{tt}$. In the Ellipsis, or $\frac{ccxx}{tt} - cc$ in Hyp. And $\overline{CF}^2 = xx + cc - \frac{ccxx}{tt}$ in-

* *Art.* 39, *and* 79.

in Ellip. or $xx - cc + \frac{ccxx}{tt}$ in Hyp. because FPC is a right-angl'd Triangle. Again, * $CP\ (x) : Ca\ (t) :: Ca\ (t) : CT = \frac{tt}{xx}$. And $\overline{PT}^2 =$ * Art. 57, and 121.
$xx - 2tt + \frac{t^4}{xx}$, and $\overline{FT}^2 = xx + cc - 2tt + \frac{t^4}{xx} - \frac{ccxx}{tt}$ in Ellip. or $xx - cc - 2tt + \frac{t^4}{xx} + \frac{ccxx}{tt}$ in Hyp. because FPT is a right-angl'd Triangle. Now the Triangles FPT, CET are similar, because the Angles at E and P are right ones, and the Angle ETC in the Ellip. common (but in the Hyperbola the Angle $ETC = PTF$) whence $\overline{FT}^2\ (xx \mp cc - 2tt + \frac{t^4}{xx} \mp \frac{ccxx}{tt}) : \overline{FP}^2\ (cc - \frac{ccxx}{tt}$ or $\frac{ccxx}{tt} - cc) ::$ $\overline{CT}^2\ \left(\frac{t^4}{xx}\right) : \overline{CE}^2 = \frac{t^4c^2}{t^4 - t^2x^2 \mp c^2x^2}$. Farther *, the Square of the Semi- * Art. 53, and 125.
conjugate, viz. $\overline{CG}^2\ (= \overline{Ca}^2 + \overline{Cb}^2 - \overline{CF}^2$ in Ellip. or $= \overline{CF}^2 + \overline{Cb}^2 - \overline{Ga}^2$ in Hyp.) is $= tt - xx + \frac{ccxx}{tt}$ in Ellip. or $xx + \frac{ccxx}{tt} - tt$ in Hyp. and $\overline{CE}^2 \times \overline{DF}^2\ (= \overline{DG}^2)$ is $= \frac{\mp t^8c^2 + t^4c^4x^2 \mp t^6c^2x^2}{t^6 \pm t^2c^2x^2 \mp t^4x^2}$, which is * * Hyp.
$= \overline{Ca}^2 \times \overline{Cb}^2 = \mp ttcc$; as it evidently appears by multiplying the Denominator by $\mp ttcc$. And therefore $Ca \times Cb = CG \times CE$, and so $4CE \times EG = 4Cb \times Ga$. W. W. D.

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AN

ANALYTICK TREATISE

OF

CONICK SECTIONS,

And their Uſe for ſolving of EQUATIONS in Determinate and Indeterminate PROBLEMS.

BOOK I.

Of the Parabola.

DEFINITIONS.

I.

IF the Rule *BC* be placed upon a Plane, together with the Square *GDO*, in ſuch manner, that *DG*, one of its Sides, lies along the Edge of that Rule; and if you take a Thread *FMO* equal in Length to *DO*, the other Side of the Square, and fix one End thereof to *O* the Extremity of the Side *DO*, and the other, in any Point *F* taken in the Plane on the ſame Side of the Rule as the Square: This being done, if you ſlide *DG*, the Side of the Square, along the Rule *BC*, and at the ſame time keep the Thread continually tight by means of the Pin *M*, with its Part *MO* cloſe to the Side of the Square *DO*: the Curve FIG. 1.

AMX, which the Pin describes by this Motion, is one Part of a Parabola.

And if the Square be turn'd about, and moves on the other side of the fix'd Point *F*, the other Part *AMZ* of the same Parabola may be describ'd after the like manner; so that the Line *XAZ* will be one and the same Curve, which is call'd a *Parabola*.

2.

The Line *BC* wherein the lower Edge of the fixed Rule *BC* touches the Plane (and the Side *DG* of the Square *GDO*) is called the *Directrix*.

3.

The fix'd Point *F*, is call'd the *Focus* of the Parabola.

4.

If the right Line *FE* be drawn from the fix'd Point *F* perpendicular to the Directrix *BC*, and meeting the Parabola in the Point *A*; then the Line *AF* infinitely produced towards *R*, is called the *Axis* of the Parabola.

5.

The Line *p* being quadruple of *AF*, is call'd the *Parameter* of the Axis.

6.

All Lines, as *MP*, drawn from Points of the Parabola perpendicular to the Axis, are called *Ordinates* to the Axis.

7.

All Lines, as *MO*, drawn from Points of the Parabola, parallel to the Axis, are called *Diameters* of the Parabola.

8.

A right Line meeting the Parabola but in one Point, and which both ways continued does not fall within the Parabola, is call'd a *Tangent* in that Point.

COROLLARY I.

1. IT follows from the Definition of the Parabola, if a right Line *MF* be drawn from *M*, any Point thereof to the Focus *F*, and another Line *MD* perpendicular to the Directrix *BC*; the right Lines *MF*, *MD*, will always be equal between themselves. For if the common Part *OM* be taken from *OD* the Side of the Square, and from the Thread *OMF*, which * is equal to it; then it is manifest, that

* Def. 1.

that the remaining Parts MD, MF, will always be equal to one another.

COROLLARY II.

2. HENCE, if any right Line KK be drawn parallel to the Directrix BC, and if from any Point M of the Parabola, there be drawn also MK perpendicular to that Line, and the right Line MF to the Focus; then the Difference or Sum (KD) of the two right Lines MF, MK, will always be the same: *viz.* the Difference, when the Point M falls below KK; and the Sum, when it falls above.

COROLLARY III.

3. IT is evident, that FE is bisected by the Parabola in the Point A. For when the Point M falls on the Point A, the Line MF falls on AF, and the Line MD on AE, which consequently will be equal to each other; since MF is always equal * to MD, let the Point M be any how taken in the Parabola. * *Art.* 1.

COROLLARY IV.

4. HENCE you may perceive, how a Parabola XAZ may be described, by having the Axis AP, (whose Origin or Vertex is A,) and its Parameter p given. For having first assumed the Parts AF, AE (on the Axis AP) on both Sides of the Origin A, each equal to the Parameter p, and drawn the indefinite Line BC from the Point E perpendicular to FE; then if the under Edge of a Rule be laid along the said Line BC, (which is the Directrix) and the Parabola XAZ be describ'd, (as is directed in *Def.* 1.) by means of the Square ODG, and a Thread FMO, equal in Length to the Side OD, (one of whose Ends is fixed to the Focus F, and the other to O, the Extremity of the same Side,) it is manifest, that this Parabola is that required.

It is also manifest, that the longer the Side (OD) of the Square, and the Thread OMF (which * must be equal to it) is, the longer likewise will the Portion of the Parabola described be; so that it may be augmented at pleasure, by augmenting equally the Side (OD) of the Square and the Thread OMF. * *Def.* 1.

COROLLARY V.

5. IF MP, an Ordinate to the Axis, be drawn from any Point M in the Parabola, together with the right Line MF to the Focus; it is manifest, that the Line MF is $= AP + AF$, because $MF = MD = AP + AE$, and * $AF = AE$. * *Art.* 3.

PROPOSITION I.

Theorem.

FIG. 1. 6. *THE Square of* MP, *any ordinate to the Axis* AP, *is equal to the Rectangle under the Parameter* p; *and the Part* (AP) *of the Axis taken from* A, *the Origin thereof to* P, *the Point of Concurrence of the Ordinate.*

We are to prove that $\overline{MP}^2 = p \times AP$.

If the given Quantity AF be called m, and the indeterminate ones AP, x; PM, y; we shall have $MF =$ * $m + x$, and $PF = x - m$ or $m - x$, according as the Point p happens below or above the Focus F. Now the right angled Triangle MPF, in both Cases, gives $\overline{MF}^2 (mm + 2mx + xx) = \overline{MP}^2 (yy) + \overline{PF}^2 (mm - 2mx + xx)$; from whence arises $4mx = yy$. Then since $p = 4m$ (by *Def.* 5.) we have also $yy = px$. *W. W. D.*

* Art. 5.

The Fundamental COROLLARY I.

7. IT is now manifest, if the Parameter of the Axis (AP) be called p; each of its Parts AP, x; and every of the correspondent Ordinates PM, y; that we shall have always $yy = px$. And since this Property agrees to all Points of the Parabola, and determines the Position thereof with respect to its Axis AP; it follows, that the Equation $yy = px$ expresses perfectly the Nature of the Parabola with regard to its Axis.

COROLLARY II.

FIG. 2. 8. IF there be drawn MP, NQ, any two Ordinates to the Axis AP, their Squares are to each other as the Parts AP, AQ of the Axis, taken from its Origin A, to P and Q, the Points of Concurrence of the same Ordinates. For * $\overline{PM}^2 : \overline{QN}^2 :: p \times AP : p \times AQ :: AP : AQ$.

* Art. 6, 7.

COROLLARY III.

9. IF the Line MPM be drawn thro' any Point P in the Axis parallel to its Ordinates; that Line will meet the Parabola in only two Points M, M, equally distant on both Sides from the Point P; for in order that the Points M and M be in the Parabola, it is

is necessary * that the Squares of each PM, taken on both Sides the Point P, be equal to the same Rectangle px. * *Art.* 7.

COROLLARY IV.

10. BEcause * yy is $= px$, it follows that the greater $AP\,(x)$ is, the more likewise will the Ordinates $PM\,(y)$ increase on both Sides the Axis AP, even to Infinity; and contrariwise, the lesser $AP\,(x)$ is, the lesser likewise will $PM\,(y)$ be: So that when $AP\,(x)$ is nothing, both the $PM\,(y)$ taken on each Side the Axis AP, will be nothing; that is, when the Point P falls in A, the two Points of Concurrence M and M will also coincide in the Point A. From whence it is manifest, * *Art.* 7.

1°. That if the Line LL be drawn parallel to the Ordinates thro' A the Origin of the Axis, it will be a Tangent in A.

2°. That the Parabola infinitely extends itself more and more on each Side the Axis AP, beginning from the Origin A; and likewise that every Parallel to the Axis, as LM, meets the Parabola but in one Point, M only, and falls within the same; since its Distance from the Axis remains always the same.

COROLLARY V.

11. IF the Line ML be drawn from any Point (M) of the Parabola, parallel to the Axis AP, meeting the Parallel (AL) to its Ordinates in L; it is manifest, if the Ordinate MP be drawn, that $AL = PM\,(y)$, and $ML = AP\,(x)\ \frac{yy}{p}$, because * $px = yy$; whence the right Lines $ML\left(\frac{yy}{p}\right)$, $ML\left(\frac{yy}{p}\right)$ taken on both Sides the Axis AP, are equal to each other, when the Points L, L, are equally distant from the Point A; and therefore if any right Line MM terminated by the Parabola be bisected by the Axis in P, it will be parallel to the Line LL, that is, it will be an Ordinate on both Sides to the Axis. For when the Parallels ML, ML, are drawn to the Axis AP, it is manifest that LL will be cut into two equal Parts in A, because MM is so cut in P. Therefore the right Lines ML, ML, will be equal between themselves, as we have prov'd; and consequently the Line MM will be parallel to LL. * *Art.* 7.

COROLLARY VI.

12. HEnce it follows that all the Perpendiculars (MPM) to the Axis, terminating on both Sides in the Parabola, † are
Art. 9. bisected in P; and that the Axis divides the Parabola into two equal Parts, similarly situate on each Side thereof. For if the two Parts of the Plane on which the Parabola is drawn, on each Side the Axis, be supposed to be folded together, it is manifest that they will exactly agree, or fall upon one another.

PROPOSITION II.

Theorem.

FIG. 3. 13. *IF any right Line* A M *be drawn from* A, *the Origin of the Axis* A P, *in either of the Angles* P A L, P A L, *made by the Axis, and the Line* L L *which is parallel to the Ordinates; I say that Line will meet the Parabola* M A M *in some other Point* M.

Take AG upon the Line AL, on either Side the Point A, equal to p the Parameter of the Axis, and draw GF parallel to the Axis, meeting the Line AM (produced if necessary) in the Point F. Moreover, take the Part AL, upon the Line AL (on the same Side of the Axis as AM is) equal to GF, and draw LM parallel to the Axis; then I say the Point M wherein LM meets AM, will be in the Parabola MAM.

For drawing MP parallel to AL the similar Triangles FGA, APM will give this proportion, FG or AL or $PM : GA :: AP : PM$; and therefore $\overline{PM}^2 = GA\,(p) \times PA$. And so the Line PM will
Art. 7. * be an Ordinate to the Axis AP. *W. W. D.*

COROLLARY I.

14. HEnce, if AP the Axis, of a Parabola, as also its Parameter, be given, and any right Line AM be drawn, from A the Origin of the Axis, in either of the Angles PAL, PAL, made by the Axis AP and the Line LL, which is parallel to the Ordinates; the Point M wherein the Line AM meets the Parabola MAM may be found.

COROL-

COROLLARY II.

15. IT is manifest * that no other Line but LAL which is parallel to the Ordinates to the Axis AP, can be a Tangent to the Parabola MAM in A the Origin of the Axis; because there is no Line but that only, which being drawn thro' the Point A, and both ways continued, but what will fall within the Parabola. * Art. 10. 13.

DEFINITIONS.

9.

If thro' any Point M in a Parabola, there be drawn a Diameter MO, an Ordinate (MP) to the Axis AP, and a right Line MT cutting the Axis produced (beyond A) in the Point T, so that AT be equal to AP: Then all right Lines, as NO, drawn from any Points in the Parabola parallel to MT, and terminating in the Diameter NO, are call'd *Ordinates* to that Diameter. FIG. 4. & 5.

10.

If the Line q be taken a third proportional to AT, MT; this Line q is call'd the *Parameter* of the Diameter MQ.

COROLLARY I.

16. IF either of the indeterminate Lines AP or AT, be called x; then it is manifest that $\overline{MT}^2 = qx$, because $AT(x) : MT :: MT : q$.

COROLLARY II.

17. BEcause MPT is a right angl'd Triangle, the Square $\overline{MT}^2 (qx)$ is $= \overline{PT}^2 (4xx) + \overline{MP}^2 * (px)$; whence, dividing by x, we have $q = 4x + p$. * Art. 7.

That is, q the Parameter of any Diameter MO, exceeds p the Parameter of the Axis by quadruple the Axis AP (x).

COROLLARY III.

18. IF the right Line MF be drawn from the Point M to the Focus F, we shall have * $MF = AP + AF$. But because (by *Def.* 5.) the Parameter of the Axis is $p = 4AF$, the Parameter of the Diameter MO will be † $q = 4AP + 4AF$. And therefore q the Parameter of any Diameter MO, will be four times the Line MF, drawn from M the Origin thereof to the Focus F. * Art. 5. † Art. 17.

PROPO-

PROPOSITION III.

Theorem.

FIG. 4, 5. 16. *THE Square of* (O N) *any Ordinate to the Diameter* M O, *is equal to the Rectangle under the Parameter* q; *and* M O, *that Part of the Diameter taken from its Origin* M *to* O, *the Point of Concurrence of the Ordinate.*

We are to prove that $\overline{ON}^2 = q \times MO$.

Draw NQ, an Ordinate, to the Axis AP, meeting the Diameter MO in the Point R, and draw OH parallel to MP; then call the given Quantities AP or AT, x; PM or RQ, y; and the indeterminate ones OR or QH, a; MO or PH, b; and then the similar Triangles TPM, ORN will give this Proportion, *viz.* $TP\,(2x) : PM\,(y) :: OR\,(a) : RN = \frac{ay}{xx}$. This being laid down:

Because (*Fig.* 4.) $NQ = RQ\,(y) - RN\left(\frac{ay}{2x}\right)$, or $RN\left(\frac{ay}{2x}\right) - RQ\,(y)$, and $AQ = AH\,(x+b) - HQ\,(a)$, when the Point N falls on the same Side of the Axis, in respect of the Diameter MO; and contrariwise (*Fig.* 5.) $NQ = RQ\,(y) + RN\left(\frac{ay}{2x}\right)$, and $AQ = AH\,(x+b) + HQ\,(a)$, when it falls on the other Side: We shall have $\overline{QN}^2 = yy \pm \frac{ayy}{x} + \frac{aayy}{4xx}$, and $AQ = x + b \pm a$, *viz.* $-$ in the first Case, and $+$ in the second. But AP* $(x) : AQ\,(x + b \pm a) :: \overline{PM}^2\,(yy) : \overline{QN}^2 = yy + \frac{byy}{x} \pm \frac{ayy}{x}$. Whence by comparing the two Values of $\overline{QN}^2$ together, we shall have this Equation, *viz.* $yy + \frac{byy}{x} \pm \frac{ayy}{x} = yy \pm \frac{ayy}{x} + \frac{aayy}{4xx}$; and striking out $yy \pm \frac{ayy}{x}$, from each Side of the Equation, dividing by yy, and multiplying by $4xx$, we shall have $\overline{OR}^2\,(aa) = 4bx$; but because the Triangles MPT, NRO are similar, therefore $\overline{PT}^2\,(4xx) : \overline{OR}^2\,(4bx) :: \overline{MT}^2$ * $(qx) : \overline{ON}^2 = bq = q \times MO\,(b)$. *W. W. D.*

* Art. 8.

* Art. 16.

A General COROLLARY. I.

20. IT is manifest (by the last *Prop.*) that what has been demonstrated in *Prop.* 1. with regard to the Axis AP, its Ordinates (PM), and Parameter p, is equally true of any Diameter MO, its Ordi-

Ordinates ($O N$) and Parameter q. Now since the 7th, 8th, 9th, 10th, 11th, 12th, 13th, 14th and 15th Articles arise from Prop. 1. and are true, whether the Angles APM be right ones, or not; it follows that if the Line $A P$ in those Articles be supposed any other Diameter instead of the Axis, whose Ordinates are the right Lines $P M$, $Q N$, and Parameter the Line p, those Articles according to this Supposition, will be still true; for their Demonstration remains always the same, and there is nothing more necessary for making this appear, but reading them over again, and using the Word Diameter for Axis.

COROLLARY II.

21. BEcause the 10th and 15th Articles are equally true, whether the Line $A P$ be the Axis, or any other Diameter, as $M O$; therefore the Line $M T$ parallel to $O N$ the Ordinates to that Diameter, touches the Parabola in M, and no other Line can touch it in that Point. FIG. 4. & 5.

Whence there can be drawn but one right Line from a given Point in a Parabola, to touch the same.

COROLLARY III.

22. HENCE it is manifest (according to Def. 9.) if ($M P$) an Ordinate to the Axis $A P$ be drawn from any Point M of a Parabola, as also another right Line $M T$ cutting the Axis (produced beyond A) so, that $A T$ be equal to $A P$; that this Line $M T$ will touch the Parabola in M. And contrariwise, if the Line $M T$ touches the Parabola in M, and $M P$ an Ordinate to the Axis be drawn; then the Parts of the Axis $A T$, $A P$ will be equal to one another.

COROLLARY IV.

23. IF you suppose in the 9th and 10th Definitions, as also in the last Proposition, the Line $A P$ to be any other Diameter instead of the Axis, whose Ordinates are the right Lines $F M$, $Q N$; that Proposition will yet appear true; because it may be demonstrated in the same Manner as before, as is evident by contemplating the 6th Figure, where the similar Triangles give the same Proportions as in the Case of the Axis. FIG. 6.

Whence it follows, 1. That the last Corollary ought still to take place, when the Line $A P$ is any other Diameter, as well as the Axis. 2. That according to that Supposition, the Diameter $M O$ may be the Axis; and so the Axis may be esteem'd such a Diameter that makes right Angles with the Ordinates thereof.

 PROP-

PROPOSITION IV.

Theorem.

FIG. 7. 24. *If* M P *an Ordinate to the Axis be drawn thro' any Point* M *in a Parabola, and also* M G *a perpendicular to the Tangent* M T *which passes through the Point* M; *I say the Part of the Axis* P G *will always be equal to the half of* p, *the Parameter thereof.*
We are to prove that P G *is* $= \frac{1}{2}$ p.

For because TPM, TMG, are right Angles, therefore TP $(2x)$:
* Art. 7. PM (y) :: PM (y) : $PG = \frac{yy}{2x} = \frac{1}{2}p$, by substituting px * the Value of yy for the same.

PROPOSITION V.

Theorem.

FIG. 7. 25. *If the right Line* M F *be drawn from the Focus* F *to any Point* M *in a Parabola, as also a Diameter* M O, *and a Tangent* T M S; *I say the Angles* F M T, O M S, *made by the Tangent* T M S, *and right Line* M F *on one Side, and the Diameter* M O *on the other, are equal to each other.*

For if the Axis AP be drawn, meeting the Tangent TMS in T,
† Art. 22. and MP an Ordinate to the Axis; we shall have † $TA + AF$ or TF
* Art. 8. $= AP + AF$ or * MF. Therefore TFM will be an Isosceles Triangle; and consequently the Angle FTM or OMS which is equal to it, will be equal to the Angle FMT. *W. W. D.*

COROLLARY.

26. HENCE it is evident that the Tangent TMS infinitely produced both ways from M, the Point of Contact, leaves the Parabola wholly next to its Focus F. And since this every where happens in whatsoever place of the Parabola the Point of Contact M be taken; it follows, that the Parabola being extended never so much, is Concave next to its Focus F.

PROP.

PROPOSITION VI.

Problem.

27. *A Diameter* A P, *together with the Tangent* L A L, *passing through* A *the Origin thereof, as also its Parameter being given; to find a Diameter* B Q, *its Origin* B, *and Parameter, which shall contain an Angle either way with its Ordinates, equal to a given Angle* K. FIG. 8, 9.

Draw the right Line *A E* through *A* the Origin of the given Diameter, making the Angle *P A E* with this Diameter equal to the given Angle *K*, and find *M* a Point * in the said *A E* (produced on the other Side of *A*, when it does not fall in either of the Angles *P A L*, *P A L*,) which may meet the Parabola. This being done, draw *Q D* through *Q* the middle Point of the Line *A M*, parallel to the Diameter *A P*, meeting the Tangent *A L* in the Point D, and bisect *Q D* in *B*. I say, the Line *B Q* is the Diameter requir'd, and the Origin thereof is the Point *B*, and its Parameter is a third proportional to *B Q*, and *Q A*. * *Art.* 14. 20.

For 1. because the Line *A M* is divided by the Diameter *B Q* into two equal Parts in the Point *Q*, the said Line will be * an Ordinate both ways to that Diameter; and since the Lines *B Q*, *A P*, are parallel to each other, the Angle *B Q A*, made by the Diameter *B Q*, and its Ordinates *Q A*, will be equal to the Angle *P A M*, equal to the given Angle *K* or its Complement to two right Angles. 2. The Point *B* the middle of the Line *Q D* will be † the Origin of that Diameter. 3. The Parameter of the Diameter *B Q* is ‖ a third Proportional to *B Q*, *Q A*. * *Art.* 11. 20. † *Art.* 22, 23. ‖ *Art.* 19.

When the given Angle *K* is not a right Angle, it is manifest that two different Lines *A E* can be drawn on each side the Diameter *A P*, making Angles with that Diameter equal to the given Angle *K*; and so we may always have two different Solutions; but you must observe that when *A E* one of the two Lines falls in the Tangent, then the Diameter *A P* itself will satisfy the Question. But when the Angle *K* is ~~not~~ a right one, since there can be drawn but one Line *A E* only, at right Angles to the Diameter; therefore in this Case the Problem will have but one Solution; and the Diameter sought will be the Axis. FIG. 8.

It must be observ'd, that the two Diameters *B Q*, *B Q*, which answer the Problem, when the Angle *K* is not a right one, are alike situate on each side the Axis *A P*, and their Parameters are equal: This appears from the Construction itself, if the given Diameter *A P* be supposed the Axis, and two Lines *A E*, *A E*, are drawn on each FIG. 10.

 Side

Side thereof. For because the Right-angled Triangles ALM, ALM, and ADQ, ADQ, are equal, and similar to one another, the Lines AD, AD; DQ, DQ; their halfs BQ, BQ; and the Ordinates QA, QA will be equal to one another; and consequently * the
* Art. 15. Parameters are so too.

Corollary.

28. HENCE it is manifest, 1. That there is but one Diameter only which is at right Angles with its Ordinates; and so a Parabola has but one Axis. 2. That there may be always found two different Diameters, making Angles with their Ordinates equal to an Angle given, provided the Angle be not a right one; and that the said two Diameters are alike situate on each side the Axis, and have equal Parameters.

Proposition VII.

Problem.

29. *A Diameter, the Parameter thereof, and a Tangent passing through its Origin, being given, to describe the Parabola by a continued Motion.*

Fig. 11. If the Diameter given be the Axis, the Parabola may be described by *Art.* 4. but if it be not, let MO be the Diameter given, and TMS the Tangent drawn through M the Origin thereof.

Assume MD in the Diameter MO (continued out beyond M,) equal to a fourth Part of the Parameter given; and draw DE of an indefinite Length perpendicular to MD. Again, draw MF, making the Angle FMT with the Tangent TMS equal to the Angle OMS, and assume MF equal to MD. This being done, if a Parabola be describ'd by Def. 1. with the Directrix DE, and the Focus F; I say, this will be that requir'd.

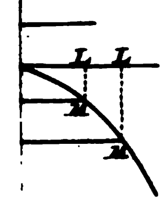
For, 1. Because the Line MO is perpendicular to the Directrix DE, it will be parallel to the Axis; and consequently will be a Diameter
* Art. 25. by Def. 7. 2. The Line TMS will be * a Tangent in M. 3. The
† Art. 28. Parameter of the Diameter MO will be † quadruple of MF.

Another Way.

Fig. 12. Let AP be the given Diameter, and LAL the Tangent passing through A the Origin thereof.

Assume

. page 12.



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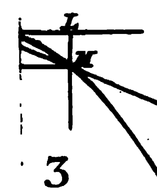

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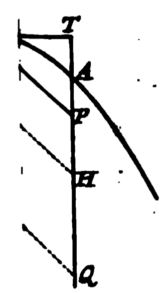
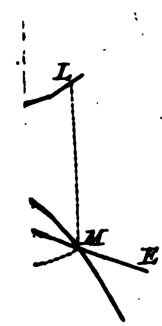




8

Assume AG in the Diameter AP (continu'd out beyond A) equal to the given Parameter, and draw an indefinite right Line DGD, making with AG the Angle AGD equal to the Angle GAL taken on the same Side; then if an indefinite right Line DM moves along the Line DG always parallel to AG; and at the same time the Extremity D thereof carries along with it the Side (DA) of the Angle DAM, equal to GAL, whose Vertex is moveable about the fixed Point A: I say, the Point M, the continual Intersection of the Line DM and the Side AM, will by this Motion describe the Parabola sought.

For if MP be drawn parallel to AL, the Lines MP, GD will be equal to one another; since the Angle APM or GAL being equal to the Angle AGD, they are equally inclin'd between the Parallels GP, DM. Now the Triangles AGD, MPA, are similar: for the Angle MPA or GAL is equal to the Angle AGD; and the Angle PMA or MAL, equal to the Angle GAD, because if the same Angle DAL be taken from each of the equal Angles GAL, DAM, the remaining Angles are equal. Therefore we shall have $AG : GD$, or $PM :: PM : AP$; and so $GA \times AP = \overline{PM}^2$; whence it is manifest *, that PM is an Ordinate to the Diameter AP, whose Origin is the Point A, the Line LAL is a Tangent in that Point, and the Line AG the Parameter to the same. *W. W. D.* * *Art.* 19. *and* 21.

If the Diameter AP be the Axis; then the Lines GD, AL will be parallel, and the Demonstration will become more easy; for it is immediately perceiv'd that GD is equal to PM, and that the Right-angled Triangles AGD, MPA are similar; whence $AG : GD$, or $PM :: PM : AP$. And therefore $AG \times AP = \overline{PM}^2$, &c. FIG. 13.

PROPOSITION VIII.

Problem.

30. *A Diameter* AP, *the Parameter thereof, and the Tangent* AL *passing thro'* A *the Origin of that Diameter, being given; to find any Number of Points of the Parabola, or (which is all one) to describe it thro' several Points.*

Assume AG in the Diameter AP (continu'd out beyond A) equal to the given Parameter, which bisect in the Point D, and draw an indefinite right Line AF perpendicular to AG; then about the Point C, taken every where in DA, produced indefinitely towards A, as a Centre, with the Radius CG, describe PF an Arc of a Circle, cutting the Diameter AP, and its Perpendicular AF, in two Points P, F: FIG. 14.

And

And draw MPM through the Point P parallel to the Tangent AL, and in this Line take the Parts PM, PM both ways, each equal to AF: and then the Points M, M will be two of those sought. And after the same manner may any Number of Pairs of Points M, M be found; through which, if a Curve Line be drawn, it will be the Parabola sought.

For all the Arcs (PF) that pass through the same Point G, and whose Centres are in the Line GA, (produced if it be necessary) will have the Lines GP for their Diameters; and therefore by the Property of the Circle we shall always have $\overline{AF}^2 = GA \times AP$. But every
* *Hyp.* PM is equal * to its Correspondent AF; and moreover, parallel to the Tangent AL, passing through A, the Origin of the Diameter AP;
† *Art.* 19, *and* 21. and consequently PM will be † an Ordinate to that Diameter; therefore the Parabola requir'd must pass through all the Points M, M, found as before directed.

It is evident, that one may err in drawing the Parts of the Parabola, that join the Points found: But this Error will not be sensible, when the Points are very near to each other. Those who have occasion to often describe the Conic Sections, commonly prefer the Way of describing them through many Points; because the Instruments that have been invented for describing them by a continued Motion, being Compound ones, are often faulty, and not exact enough in Practice.

Another Way.

FIG. 15. Draw the indefinite right Line LE through L, any Point in the Tangent AL, parallel to the Diameter AP; and in this Line and the Diameter AP (continued out beyond A) assume at pleasure the equal Parts LE, EE, EE, &c. AF, FF, FF, &c. and take the Point M in LE, so that LM be a third Proportional to the Parameter of the Diameter AP given, and AL the Part of the Tangent. Then if the Lines AE, AE, AE, &c. MF, MF, MF, &c. be drawn from the Points A and M: I say, the Points of Intersection N, N, N, &c. of every AE, and its Correspondent MF, will be all in the Parabola requir'd.

For if the Lines MP, NQ are drawn through the Points M and N, parallel to the Tangent AL, and you call AP, x; PM or AL, y; AQ, u; QN, z; the similar Triangles NQA, ALE and MPF; NQF will give these two Proportions, *viz.* $QN\ (z) : QA\ (u) :: AL\ (y) : LE$ or $AF = \frac{uy}{z}$. And $MP\ (y) : PF$ or $PA + AF\ (x + \frac{uy}{z}) :: NQ\ (z) : QF$ or $QA + AF\ (u + \frac{uy}{z})$. And multiplying the

 Means

Means and Extremes, we shall ~~from~~ form this Equation $uy + \frac{uyy}{z} = xz + uy$; and striking out uy from both Sides of the Equation, and multiplying by z, we shall have $uyy = xzz$, which may be reduced to this Proportion $AP\,(x) : AQ\,(y) :: \overline{MP}^2\,(yy) : \overline{NQ}^2\,(zz)$. But by Construction, the Square of AL or PM is equal to the Rectangle under AP, the Part of the given Diameter, and the Parameter thereof. And therefore the Line PM will be * an Ordinate to the Diameter AP; and so QN will be † another. Consequently the Point N will be one Point of the Parabola, that falls on one Side of the Diameter AP: And to find the Points on the other Side, you need only take the equal Parts LE, EE, &c. AF, FF, &c. in the indefinite right Lines LE, AF, on the other Side of the Points L, A.

* *Art.* 19. *and* 21.

† *Art.* 8. *and* 20.

If instead of the Parameter of the Diameter AP, which is here suppos'd to be given, we have M one of the Points in the Parabola, which often happens: Then you must draw the indefinite Line LE thro' that Point parallel to the Diameter AP, and proceed afterwards as above.

The End of the First Book.

BOOK

BOOK II.

Of the Ellipsis.

Definitions.

1.

Fig. 16. IF F, f be two Points, or Nails fix'd in a Plane, and a Thread FMf be put about them, whose Length must be more than the Distance Ff; and if then you put the Pin M to the Thread, so as always to keep the same strain'd tight; and move the Pin round these two Points, till it be come to the same Place from whence it went; the said Pin, by this Motion, will describe a Curve, which is called an *Ellipsis*.

2.

The Points F, f, are called the *Foci* of the Ellipsis.

3.

The Line Aa, which passes thro' the two Foci F, f terminating both ways in the Ellipsis, is called the *first* or *great Axis* [or *transverse Axis*.]

4.

The Point C, which divides the first Axis Aa in half, is called the *Centre* of the Ellipsis.

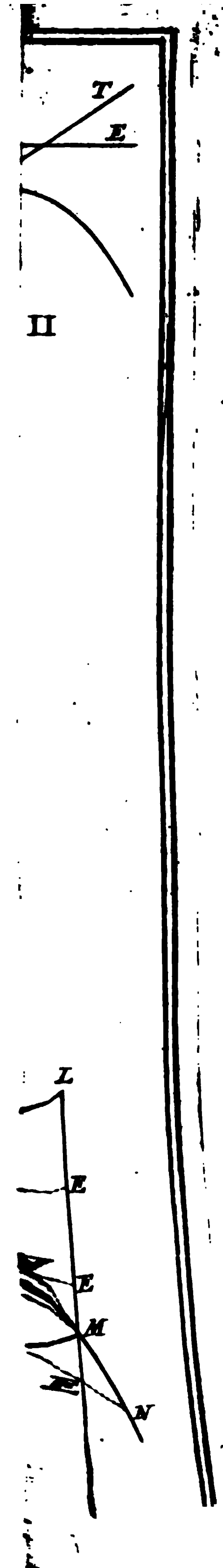
5.

The Line Bb drawn through the Centre C, perpendicular to the first Axis Aa, and terminating both ways in the Ellipsis, is called the *second* or *small Axis* [or Conjugate Axis.]

6.

The two Axes Aa, Bb, are called together, *Conjugate Axes*: So that the first Axis Aa is said to be a Conjugate to the second Bb; and reciprocally the second Bb, a Conjugate to the first Aa.

7. The


T
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7.

The Lines MP, MK drawn from Points (M) of the Ellipsis, parallel to one of the Axes, and terminated by the other, are call'd Ordinates to that other Axis: So MP is an Ordinate to the Axis Aa, and MK to the Axis Bb.

8.

A third Proportional to the two Axes, is called the *Parameter* of that which is the first Term of the Proportion: So if it be made as the first Axis Aa to the second Bb, so is the second Bb to a third Proportional p; this Line p will be the Parameter of the first Axis.

9.

All right Lines passing through the Centre C, and terminated both ways by the Ellipsis, are called *Diameters*.

10.

A right Line meeting the Ellipsis but in one Point, and being both ways continued does not fall within the same, is called a *Tangent* in that Point.

SCHOLIUM.

31. IF the two Foci F, f, and the Centre C be supposed to be united in one Point, it is manifest then, that the Ellipsis will be changed into a Circle, having the right Line CM for the Radius thereof, equal to one half of the Thread CMC, put about the Point C, the Centre of the Circle. Hence a Circle may be consider'd as a particular Species of an Ellipsis, wherein the Distance of the Foci is nothing: And therefore, whatever in the following Treatise is demonstrated of Ellipses, be their focal Distance what it will, may be also apply'd to a Circle, by supposing that Distance to become nothing. FIG. 17.

COROLLARY I.

32. IT follows from the first Definition, that if the right Lines MF, Mf be drawn from any Point M of the Ellipsis; their Sum will be always the same. FIG. 16.

COROLLARY II.

33. IT is manifest, that when the Point M falls in A, the Line MF will become AF, and Mf the Line Af: It is moreover evident, that when the Point M falls in a, the Line MF will become

aF, and Mf will become the Line af. And therefore we shall have $AF + Af$, or $2\,AF + Ff = aF + af$, or $2\,af + fF$; and consequently $AF = af$. Whence it follows:

1. That the Sum of the two right Lines MF, Mf, is always equal to the great Axis Aa, since $Mf + MF = Af + AF = Af + fa$.

2. That the focal Distance Ff is divided into two equal Parts by the Centre C, because $CA - AF$, or $CF = Ca - af$, or Cf.

COROLLARY III.

34. IF the right Lines BF, Bf be drawn from B, the Extremity of the second Axis Bb to the two Foci F,f; it is then manifest, that the right-angled Triangles BCF, BCf are equal to each other; and so BF the Hypothenuse of the one, equal to Bf the Hypothenuse of the other; therefore BF, or $Bf = CA$, or Ca, because * $BF + Bf = Aa$. And after the same manner we prove, that Fb, or fb is $= CA$ or Ca. Whence it appears:

* Art. 33.

1. The second Axis Bb is divided into two equal Parts by the Centre C; for the right-angled Triangles FCB, FCb are equal; because their Hypothenuses FB, Fb are equal, and the Side FC is common.

2. The second Axis Bb is always less than the first one Aa; because BC, the half thereof, being one of the Sides of the right-angled Triangle BCF, will be less than the Hypothenuse BF, which is equal to CA, the half of the first Axis Aa.

3. If a Circle be described about B, one of the Extremes of the second Axis Bb, with the Radius BF, equal to CA, the half of the first or great Axis Aa; then that Circle will cut the great Axis in the two Points F,f, which will be the two Foci of the Ellipsis.

COROLLARY IV.

35. THE same Things being premised, if CA or BF be called t; and CF, m; then the right-angled Triangle BCF, will give $\overline{BC}^2 = tt - mm$. But $AF = t - m$, and $Fa = t + m$, and therefore $AF \times Fa = tt - mm$. Whence it is manifest, that the Square of CB, half of the little Axis Bb, is equal to the Rectangle under AF, and Fa, the two Parts of the great Axis between A, a, the Ends thereof, and the Focus F.

COROLLARY V.

36. HEnce it will be easy to describe an Ellipsis, whose two Axes Aa, Bb, are given: For if the two Foci F,f be found in the great Axis, and you put the Thread FMf, whose Length is equal to the

the great Axis, about them; and then describe an Ellipsis, according to the Directions of *Def.* 1. it is evident, that the Ellipsis describ'd will be that requir'd.

PROPOSITION I.

Theorem.

37. *IF the Ordinate* MP *be drawn to the first or great Axis* Aa, *and the Part* AD *be taken upon that Axis equal to* MF; *I say,* CA : CF :: CP : CD. FIG. 16.

Call (as before) the given Quantities CA, CF, t and m; and the Indeterminate ones CP, PM, x and y; and CD the unknown one z: Now there may happen two Cases.

Case 1. When the Point P falls above the Centre C. Because PF is always less than Pf; therefore MF or AD will be less than Mf or aD; and so AD or $MF = t - z$, aD or $Mf = t + z$, $FP = m - x$ or $x - m$ (according as the Point P falls below or above the Focus F), and $Pf = x + m$. But the right-angled Triangles MPF, MPf, give $tt - 2tz + zz = yy + mm - 2mx + xx$, and $tt + 2tz + zz = yy + mm + 2mx + xx$. And by substracting the former Equation from the latter, we shall have $4tz = 4mx$; and consequently $CD\ (z) = \frac{mx}{t}$.

Case 2. When the Point P falls below the Centre C, because PF is always greater than Pf, it is manifest, that MF or AD will be greater than Mf or aD: therefore AD or $MF = t + z$, aD or $Mf = t - z$, $PF = x + m$, $Pf = x - m$, or $m - x$ (according as the Point P falls below or above the Focus f) but the right-angled Triangles MPF, MPf, give $tt + 2tz + zz = yy + mm + 2mx + xx$, and $tt - 2tz + zz = yy + mm - 2mx + xx$. And by substracting this last Equation from the former, we shall have $4tz = 4mx$, and consequently $CD\ (z) = \frac{mx}{t}$. Whence in both Cases we have $CA\ (t) : CF\ (m) :: CP\ (x) : CD\ (z)$. *W. W. D.*

COROLLARY.

38. HENCE it is manifest, if the given Quantities CA or Ca be called t; CF or Cf, m; and the indeterminate one CP, x: we shall always have $MF = t - \frac{mx}{t}$, and $Mf = t + \frac{mx}{t}$, when the

 Point

Point P falls above the Centre C: And on the contrary, when the Point P falls below it, we shall have $MF = t + \frac{mx}{t}$, and $Mf = t - \frac{mx}{t}$.

PROPOSITION II.

Theorem.

39. *THE Square of* MP, *any Ordinate to the Axis* Aa, *is to the Rectangle under* AP *and* Pa, *the Parts of that Axis, as the Square of its Conjugate Axis* Bb, *to the Square of the Axis* Aa.

We are to prove, that $\overline{PM}^2 : AP \times Pa :: \overline{Bb}^2 : \overline{Aa}^2$.

The same being suppos'd, as in the foregoing Article, if $\frac{mx}{t}$ $(=z)$ be put for z in the Equation $tt \pm 2tz + zz = yy + mm$

* Art. 37. $\pm 2mx + xx$, which was found * by means of the right-angled Triangle MPF, we shall always form this, *viz.* $ttyy = t^4 - ttxx - mmtt + mmxx$, which being reduced to a Proportion, gives $\overline{PM}^2$ (yy):

† Art. 35. $AP \times Pa$ $(tt - xx) :: \overline{BC}^2$ † $(tt - mm) : \overline{CA}^2$ $(tt) :: \overline{Bb}^2 : \overline{Aa}^2$. *W. W. D.*

COROLLARY I.

40. IF any Ordinate MK be drawn to the other Axis Bb, which I call $2c$, it is manifest that $MK = CP$ (x), and $CK = PM$

* Art. 39. (y). But * $\overline{PM}^2$ $(yy) : AP \times Pa$ $(tt - xx) :: \overline{Bb}^2$ $(4cc) : \overline{Aa}^2$ $(4tt)$. And therefore $4cc = xx = 4cctt - 4ttyy$; and so we have this proportion $\overline{MK}^2$ $(xx) : BK \times Kb$ $(cc - yy) :: \overline{Aa}^2$ $(4tt) : \overline{Bb}^2$ $(4cc)$.

That is, the Square of MK any Ordinate to the Axis Bb, is to the Rectangle under BK and Kb, the Parts of that Axis, as the Square of its Conjugate Axis Aa, is to the Square of the Axis Bb.

The Fundamental COROLLARY I.

FIG. 18, and 19. 41. IF one of the Axes, as Aa, be call'd $2t$; its Conjugate Bb, $2c$; the Parameter thereof p; every of the Ordinates PM, y; and each of its Parts (CP) contain'd between the Centre and the Points of Concurrence of the Ordinates, x; we shall always have *

* Art. 39. $\overline{PM}^2$ $(yy) : AP \times Pa$ $(tt - xx) :: \overline{Bb}^2$ $(4cc) : \overline{Aa}^2$ $(4tt) :: p : Aa$ $(2t)$

$(2t)$. Because by the Definition of a Parameter, $Aa\,(2t) : Bb\,(2c) :: Bb\,(2c) : p$, therefore $p = \frac{4cc}{2t}$. Whence if the Means and Extremes of the proportion $yy : tt - xx :: 4cc : 4tt$, are first multiply'd into one another, and afterwards those of this $yy : tt - xx :: p : 2t$. We shall have $yy = cc - \frac{ccxx}{tt}$, and $yy = \frac{1}{2}pt - \frac{pxx}{2t}$. Now since this Property equally agrees to all the Points of the Ellipsis, and determines the Position thereof with respect to the two Conjugate Axes Aa, Bb; it follows that the Equation $yy = cc - \frac{ccxx}{tt}$, or $yy = \frac{1}{2}pt - \frac{pxx}{2t}$, expresses the Nature of the Ellipsis with regard to the Axes thereof.

COROLLARY III.

42. If there be drawn MP, NQ, any two Ordinates to the Axis Aa; their Squares will be to one another as ($AP \times Pa$, $AQ \times QA$), the Rectangles under the Parts of the Axis made by the Points of Concurrence of the said Ordinates; for * $\overline{Bb}^2 : \overline{Aa}^2 :: \overline{PM}^2 : AP \times Pa :: \overline{QN}^2 : AQ \times Qa$. And therefore $\overline{PM}^2 : \overline{QN}^2 :: AP \times Pa : AQ \times Qa$. * Art. 39.

COROLLARY IV.

43. If MM be drawn from any Point P in one of the Conjugate Axes Aa, parallel to the other Axis Bb; that Line will meet the Ellipsis in only two Points M, M, equally distant on each side of the Point P, and not more. For that the Points M, M, be in the Ellipsis, it is * necessary that the Squares of $PM\,(y)$ taken on both Sides of the Axis Aa, be each equal to the same Quantity, *viz.* $cc - \frac{ccxx}{tt}$. * Art. 41.

COROLLARY V.

44. Because * yy is $= cc - \frac{ccxx}{tt}$, it follows, that the more $CP\,(x)$ taken both ways from C the Centre, increases, the more does both the Ordinates $PM\,(y)$ taken on both Sides of either of the Axes Aa, diminish; so that when $CP\,(x)$ is equal to CA, or $Ca\,(t)$, both the Ordinates $PM\,(y)$ will then become nothing: And contrariwise, the more $CP\,(x)$ diminishes, the more will both the Ordinates $PM\,(y)$ taken on each side of the Axis Aa increase; so that when $CP\,(x)$ becomes * Art. 41.

becomes nothing, both the Ordinates PM (y), each of which then is CB or Cb (c) will be the greatest of the Ordinates. Whence is is manifest,

1. That if right Lines be drawn through B, b, the Ends of one of the Conjugate Axes parallel to the other; those Lines will touch the Ellipsis in them Points.

2. That the Ellipsis recedes more and more from A the Extremity of either of its Axes Aa, until it meets the Conjugate Axis Bb; after which it continually accedes to the same Axis Aa, until it meets the same in a, the other End thereof.

COROLLARY VI.

* Art. 41. 45. BEcause * $yy = cc - \frac{ccxx}{tt}$, it follows, that if the Points P, P, be taken equally distant both ways from the Centre C; the Ordinates PM, PM, will be equal. And therefore, if any right Line MM terminating in the Ellipsis, be cut into two equal Parts, by one of the Conjugate Axes Bb, in the Point K, not being the Centre, that right Line will be parallel to the other Axis Aa: For draw MP, MP, parallel to the Axis Bb, and then PP will be bisected in C, because MM is so divided in K; therefore the Ordinates PM, PM will be equal: And so the right Line MM will be parallel to the Axis Aa.

COROLLARY VII.

46. IF the two Parts of the Plane whereon the Ellipsis is drawn, on each side of either of the Axis, as Bb, be supposed to be folded together; it is manifest that the two Semi-Ellipses BAb, Bab, will exactly fall on each other, [*or Coincide*] *viz.* the Points A, M, &c. on a, M, &c. because * all the Perpendiculars Aa, MM, &c. to that Axis, are bisected in the Points C, K, &c. Whence it appears that the Ellipsis is cut by the two Axes into four equal and uniform Parts, not at all differing but in Situation.

* Art. 45.

PROPOSITION III.

Theorem.

FIG. 20. 47. *IF any right Line* AM *be drawn thro'* A *one of the Extremities of one of the Axes* Aa, *in one of the Angles* aAL, aAL, *made by that Axis, and the Line* LAL *parallel to* Bb *its Conjugate Axis; I say the said Line will meet the Ellipsis in some other Point* M.

Take

Take AG, on either side the Point A, in the Line AL, equal to p the Parameter of the Axis Aa, and draw GF parallel to the Axis, meeting the Line AM (produced, if necessary) in the Point F: Moreover, assume AL in the Line AL, (on the same Side as the Line AM falls with respect to the Axis Aa) equal to GF, and draw the right Line aL through a the other Extremity of the Axis Aa; I say the Point M wherein this Line cuts the Line AM, is in the Ellipsis MAM.

For draw MP parallel to AL, and call the known Lines Aa, AG, $2t$, p; and GF or AL, a; and the unknown ones CP, x; PM, y; then the similar Triangles AGF, MPA and LAa, MPa, give this proportion, $AG(p):GF(a)::MP(y):AP(t\pm x)=\frac{ay}{p}$. And $AL(a):Aa(2t)::PM(y):aP(t\mp x)=\frac{2ty}{a}$. And consequently we have always $AP\times Pa\,(tt-xx)=\frac{2tyy}{p}$, let the Point P fall either above or below the Centre C; and therefore $yy=\frac{1}{2}pt-\frac{pxx}{2t}$. Whence the Line PM will be * an Ordinate to the Axis Aa; and so the Point M will be in the Ellipsis MAM. W. W. D. * Art. 41.

COROLLARY I.

48. HENCE if Aa one Axis of an Ellipsis MAM, as also p the Parameter thereof be given, and any right Line AM be drawn through A one of the Extremities of that Axis, in either of the Angles aAL, aAL made by that Axis, and the Line LAL which is parallel to the Conjugate Axis Bb: Then it appears how the Point M wherein the Line AM meets the Ellipsis MAM may be found.

COROLLARY II.

49. HENCE it is evident that there is but one Line LAL parallel to the Axis Bb, that can touch the Ellipsis MAM in the Point A, one of the Ends of Aa the Conjugate Axis thereto; because no Line but that only, which passing through the Point A, and being continued both ways, but what will fall within the Ellipsis, and meet it in another Point.

PRO-

PROPOSITION IV.

Theorem.

FIG. 20. 50. *ALL Diameters, as* M C m *are cut into two equal Parts by the Centre* C, *and meet the Ellipsis but in two Points* M, m.

Draw the Ordinate MP, and assume Cp equal to CP, and on the Point p raise the perpendicular pm terminating in the right Line MCm; then it is manifest that the Triangles CPm, Cpm are similar and equal, and so CM is equal to Cm, and PM to pm. And those Ordinates that are equally distant from the Centre C on both Sides thereof, are * equal to one another. But PM is an Ordinate; therefore pm will also be an Ordinate, and consequently the Point m will be in the Ellipsis.

* Art. 45.

Moreover, if a right Line parallel to the Axis Bb, be supposed to move from C towards A, it is manifest that the Part of that Line included within the Angle ACM, will continually increase as CP does. And contrariwise the Part of that Parallel included between the Quadrant of the Ellipsis AMB, and the Axis CA; that is, the Ordinate PM, will * continually decrease; and so the right Line CM, passing through the Centre, meets the Ellipsis in one Point M only, on the same Side the Axis; and the same is to be understood of the Point m taken on the other side. Whence, &c.

* Art. 44.

DEFINITIONS.

11.

FIG. 21, and 22. If through any Point M of an Ellipsis, be drawn a Diameter MCm, an Ordinate MP to either of the Axes, as Aa, and a right Line, in such manner that CT be a third proportional to CP and CA; the Diameter SCs which is parallel to MT, is call'd the *Conjugate Diameter* to the Diameter Mm. And contrariwise, the Diameter Mm is said to be a Conjugate to the Diameter Ss: So that both of them together are call'd *Conjugate Diameters*.

12.

All right Lines drawn from Points of the Ellipsis parallel to one of the said two Diameters, and terminating in the other, are call'd *Ordinates* to that other Diameter. So NO which is parallel to the Diameter Ss, is an Ordinate to Mm the Conjugate Diameter thereto.

13. A third

13.

A third Proportional to two Conjugate Diameters, is call'd the *Parameter* of the first Term of the Proportion: So a third Proportional to Mm, Ss, is call'd the *Parameter* of the Diameter Mm.

COROLLARY.

51. IF the given Line CA be called t; and the Indeterminate ones CP, PT, x, s; it is manifest (by *Def.* 11.) that $CT\ (x+s = \frac{tt}{x}$; and so $sx = tt - xx = AP \times Pa$.

PROPOSITION V.

Theorem.

52. *IF two Ordinates,* MP, SK, *to the Axis* Aa, *be drawn thro'* M, S, *the Ends of the Conjugate Diameters* Mm, Ss; *I say,* CK, *the Part of the Axis between the Centre and the Point of Concurrence of one of the Ordinates* SK, *is a mean Proportional between* AP, Pa, *the two Parts of the Axis made by the Concurrence of the other Ordinate* MP.

We are to prove, that $\overline{CK}^2 = AP \times Pa$.

Call the known Lines CA, t; CP, x; PT, s; and the unknown one CK, m; then we shall have $AP \times Pa = tt - xx = {}^{*}sx$, and $AK \times Ka = tt - mm = sx + xx - mm$, by putting $xx + sx$ for tt, which it is equal to. Now, by the Property of the Ellipsis, † $AP \times Pa\ (sx) : AK \times Ka\ (sx + xx - mm) :: \overline{PM}^2 : \overline{KS}^2 :: \overline{TP}^2\ (ss) : \overline{CK}^2\ (mm)$. By the Similarity of the Triangles TPM, CKS; whence by multiplying the Means and Extremes, and orderly transposing, there arises $\overline{CK}^2\ (mm) = \frac{sxx + ssx}{x+s} = sx = AP \times Pa$. *W. W. D.* * *Art.* 51. † *Art.* 42.

COROLLARY.

53. BEcause $\overline{CK}^2 = tt - xx$, it follows, that $\overline{CA}^2 - \overline{CK}^2$, or $AK \times Ka = xx$. But * $\overline{CA}^2\ (tt) : \overline{CB}^2\ (cc) :: AK \times Ka\ (xx) : \overline{SK}^2 = \frac{ccxx}{tt}$. And $\overline{CA}^2\ (tt) : \overline{CB}^2\ (cc) :: AP \times Pa\ (tt - xx) : \overline{PM}^2 = cc - \frac{ccxx}{tt}$. Moreover, because the Triangles CPM, CKS are * *Art.* 41.

right-angled, we have the Square $\overline{CM}^2$ or $\overline{CP}^2 + \overline{PM}^2 = xx + cc - \frac{ccxx}{tt}$, and the Square $\overline{CS}^2$ or $\overline{CK}^2 + \overline{KS}^2 = tt - xx + \frac{ccxx}{tt}$. Therefore $\overline{CM}^2 + \overline{CS}^2 = tt + cc$.

That is, the Sum of the Squares of any two Conjugate Diameters *Mm*, *Ss*, is equal to the Sum of the Squares of the two Axes *Aa*, *Bb*.

PROPOSITION VI.

Theorem.

54. *THE Square of* ON, *any Ordinate to the Diameter* Mm, *is to the Rectangle* MO, Om, *under the Parts of that Diameter, as the Square of* Ss, *the Conjugate Diameter to it, to the Square of the said Diameter* Mm.

We are to prove, that $\overline{ON}^2 : MO \times Om :: \overline{Ss}^2 : \overline{Mm}^2$.

Draw *NQ*, *OH* parallel to the Axis *Bb*, and *OR* parallel to *Aa*, the Conjugate thereto, meeting the Ordinate *NQ* (produced, if necessary) in the Point *R*; then call the given Lines *CP*, x; *PM*, y; *CA*, t; *PT*, s; and the Indeterminate ones *HQ* or *OR*, a; *CH*, b; and the similar Triangles *CPM*, *CHO*, and *MPT*, *NRO*, will give these two Proportions $CP\,(x) : PM\,(y) :: CH\,(b) : HO$ or $RQ = \frac{by}{x}$. And $TP\,(s) : PM\,(y) :: OR\,(a) : RN = \frac{ay}{s}$.

And because (*Fig.* 21.) *NQ* is always the Difference of $RQ\left(\frac{by}{x}\right)$ $RN\left(\frac{ay}{s}\right)$, and *CQ* the Sum of $CH\,(b)$, $HQ\,(a)$, when the Point *N* falls between the Points *M*, *S*, or *m*, *s*; and on the contrary, (*Fig.* 22.) *NQ* always the Sum of *RQ*, *RN*, and *CQ*, the Difference of *CH*, *HQ*, when the Point *N* falls otherwise, we have $\overline{NQ}^2 = \frac{bbyy}{xx} \mp \frac{2abyy}{sx} + \frac{aayy}{ss}$, and $\overline{CQ}^2 = aa \mp 2ab + bb$; viz. $-\frac{2abyy}{sx}$, and $+2ab$ in the first Case, and $+\frac{2abyy}{sx}$, and $-2ab$ in the second. But * $AP \times Pa\,(tt - xx) : AQ \times Qa$, or $\overline{CA}^2 - \overline{CQ}^2\,(tt - aa \pm 2ab - bb) :: \overline{PM}^2\,(yy) : \overline{QN}^2 = \frac{ttyy - aayy + 2abyy - bbyy}{tt - xx}$. And by comparing the two Values of $\overline{NQ}^2$ together, this Equation will be had $\frac{bbyy}{xx} \pm \frac{2abyy}{sx} + \frac{aayy}{ss} = \frac{ttyy - aayy \pm 2abyy - bbyy}{tt - xx}$, and striking out the Term $\pm \frac{2abyy}{sx}$, from one

* Art. 42.

 Side

Side of the Equation, and the Term $+\frac{2abyy}{tt-xx}$ from the other Side (these two Terms being equal, since, by *Art.* 51. $sx = tt - xx$) and dividing by yy, there will be had $\frac{bb}{xx} + \frac{aa}{ss} = \frac{tt-aa-bb}{tt-xx}$.

And multiplying by xx, and transposing bb, we shall have $\frac{aaxx}{ss}$, or $\frac{aax^4}{ssxx} = \frac{ttxx-aaxx-bbtt}{tt-xx}$; and again multiplying the first Member by $ssxx$, and the second by the Square of $tt-xx$, the Value of sx (which is done in only multiplying the Numerator by $tt-xx$) we shall have $aax^4 = t^4xx - aattxx - bbt^4 - ttx^4 + aax^4 + bbttxx$; and striking out aax^4 from both Sides, transposing of $aattxx$, and dividing by $ttxx$, there will be had $\overline{HQ}^2$ or $\overline{OR}^2\,(aa) = tt - xx + bb - \frac{bbtt}{xx}$.

Now if the Semidiameter CM or Cm be called z; by the Similarity of the Triangles CPM, CHO, we shall have the following Proportion $CP\,(x) : CM\,(z) :: CH\,(b) : CO = \frac{bz}{x}$. Therefore $MO \times Om = zz - \frac{bbzz}{xx}$. But the similar Triangles ORN, CKS give this Proportion $\overline{ON}^2 : \overline{CS}^2 :: \overline{OR}^2\,\left(tt - xx + bb - \frac{bbtt}{xx}\right) : \overline{CK}^2$ * * *Art.* 52.
$(tt - xx) :: MO \times Om\,\left(\frac{xxzz - bbzz}{xx}\right) : \overline{CM}^2\,(zz.)$ Since the same Product arises by multiplying the Means and the Extremes. And therefore $\overline{ON}^2 : MO \times Om : \overline{CS}^2 : \overline{CM}^2$ * $:: \overline{Ss}^2 : \overline{Mm}^2$. *W. W. D.* * *Art.* 50.

A General COROLLARY. I.

55. HENCE it is manifest, that what has been demonstrated in *Prop.* 2. with respect to the two Axes Aa, Bb, is equally true of any two Conjugate Diameters Mm, Ss. And since the 40th, 41st, 42d, 43d, 44th, 45th, 47th, 48th and 49th Articles, arising from *Prop.* 2. are true, whether the Angle ACB be right or not; it follows, that if the Lines Aa, Bb, are suppos'd in those Articles, instead of the two Axes, any two Conjugate Diameters, those Articles will still be true according to this Supposition: For their Demonstration will always remain the same; and there is nothing more requir'd to make this appear, but reading them over again, and every where using the word *Diameter* for *Axis*.

COROLLARY II.

56. BEcause the 44th and 49th Articles are equally true, whether the Lines Aa, Bb, are any two Conjugate Diameters, as Mm, Ss, as well as the Axes; therefore the Line MT drawn from the Point M, one End of any Diameter Mm, parallel to Ss the Conjugate Diameter thereto, touches the Ellipsis in M; and no other Line but that can touch the Ellipsis in that Point.

Whence it appears, that there can but one Line be drawn from a given Point in an Ellipsis, to touch the same in that Point.

COROLLARY III.

57. HENCE it is evident, by *Def.* 11. that if an Ordinate MP, be drawn from any Point M of an Ellipsis, to either of the Axes Aa; and you take CT, towards the Point P, a third Proportional to CP, CA, and draw the right Line MT; this Line MT will touch the Ellipsis in M. And contrariwise, if the Line MT touches the Ellipsis in M, and the Ordinate MP be drawn to the other Axis Aa, the Parts of that Axis CP, CA, CT, will be in a continual Geometrick Proportion.

COROLLARY IV.

58. IN the 11th, 12th, and 13th Definitions and in the two last Propositions, if you suppose that the Lines Aa, Bb, instead of the Axes, are any two conjugate Diameters, those Propositions will still be true, because they may be demonstrated as before: as is evident by contemplating the 23d Figure, wherein the similar Triangles give the same Proportions as in the Case of the Axes.

Whence it follows, 1. That the last Corollary ought still to take place, when the Line Aa is any Diameter, as well as the Axis. 2. That the Conjugate Diameters Mm, Ss, may be the two Axes according to that Supposition; and so the two Axes may be esteemed as two Conjugate Diameters, being at right Angles with each other.

PROPOSITION VII.

Theorem.

FIG. 24. 59. *IF from any Point of an Ellipsis, whose Centre is* C, *there be drawn the Ordinate* MP *to* Aa *one of the Axes, and* MG *perpendicular to the Tangent* MT *passing through the Point* M; *I say* CP *will always have*

have the same Proportion to PG, *as the Axis* Aa *has to its Parameter.*

For if the Semi-Axis CA or Ca be called t; and the indeterminate Lines CP, x; PM, y; we shall have * $CT = \frac{tt}{x}$; and therefore $PT = \frac{tt - xx}{x}$. But the Right-angl'd similar Triangles TPM, MPG, give this proportion, $TP\left(\frac{tt-xx}{x}\right) : PM(y) :: PM(y) : PG\left(\frac{xyy}{tt-xx}\right)$. Hence we may get this Proportion $CP(x) : PG\left(\frac{xyy}{tt-xx}\right) :: AP \times Pa$ $(tt - xx) : \overline{PM}^2(yy)$. Because by multipling the Means and Extreemes, the same Product xyy will arise, but the Rectangle $AP \times Pa$ is * to the Square $\overline{PM}^2$, as the Axis Aa to its Parameter. Wherefore, &c.

* Art. 57.

* Art. 43.

PROPOSITION VIII.

Theorem.

60. IF a Tangent TMS be drawn through any Point M of an Ellipsis, as also the right Lines MF, Mf, to the two Foci F, f: I say the Angles FMT, fMS, made both ways by these two Lines, with the Tangent TMS, are equal to one another. FIG. 25.

For draw FD, fd, perpendicular to the Tangent; also draw the first Axis Aa meeting it in T, and the Ordinate MP to that Axis, and call CA or Ca, t; CF or Cf, m; and CP, x; then we shall have MF * $\left(t - \frac{mx}{t}\right) : Mf\left(t + \frac{mx}{t}\right) :: TF$, or CT * $\left(\frac{tt}{x}\right) - CF(m) :$ Tf or $CT\left(\frac{tt}{x}\right) + Cf\ (m.)$ Because in multiplying the Means and Extremes, the same Product arises. But the similar Triangles TFD, Tfd, give this Proportion, $TF : Tf :: FD : fd$. And so MF the Hypothenuse of the right-angled Triangle MDF will be to Mf, the Hypothenuse of the right-angled Triangle Mdf, as the Side DF is to the Side df: and consequently these two Triangles will be similar: wherefore the Angles FMD, fMd, or FMT, fMS, which are opposite to the Homologous Sides DF, df, will be equal to one another. W. W. D.

* Art. 38.
* Art. 57.

COROL-

COROLLARY.

FIG. 26. 61. HENCE it is manifest, that the Tangent TMS being both ways infinitely produced from the Point of Contact M, leaves the Ellipsis entirely next to its two Foci F, f. And since this is always so, let the Point M be where it will in the Ellipsis, it follows, that the Ellipsis will be Concave quite round about the two Foci thereof, and consequently also about its Centre.

PROPOSITION IX.

Theorem.

FIG. 26. 62. *IF* DAE *be drawn thro'* A, *one End of any Diameter* A a, *parallel to* B b, *the Conjugate Diameter thereto, meeting any two other Conjugate Diameters* Mm, Ss, *in the Points* D, E; *I say, the Rectangle under* DA, AE, *is equal to the Square of* CB, *the one half of the Diameter* Bb.

We are to prove, that $DA \times AE = \overline{CB}^2$.

Through M, S, the Ends of the Conjugate Diameters Mm, Ss, draw the Ordinates MP, SK to the Diameter Aa, and call CA, t; CB, c; and CP, x; PM, y; then we shall* have $\overline{CK}^2 = AP \times Pa = tt - xx$; and consequently $AK \times Ka$, or $\overline{CA}^2 - \overline{CK}^2 = xx$. But* $\overline{BC}^2 (cc) : \overline{CA}^2 (tt) :: \overline{MP}^2 (yy) : AP \times Pa$, or $\overline{CK}^2 = \frac{ttyy}{cc}$. And $\overline{CA}^2 (tt) : \overline{CB}^2 (cc) :: AK \times Ka (xx) : \overline{KS}^2 = \frac{ccxx}{tt}$. And by extracting the square Root, we get $CK = \frac{ty}{c}$, and $KS = \frac{cx}{t}$. But the similar Triangles CPM, CAD, and CKS, CAE, give these Proportions $CP (x) : PM (y) :: CA (t) : AD = \frac{ty}{x}$. And $CK \left(\frac{ty}{c}\right) : KS \left(\frac{cx}{t}\right) :: CA (t) : AE = \frac{ccx}{ty}$. Therefore $DA \times AE = cc = \overline{BC}^2$. W.W.D.

* Art. 52.

* Art. 54.

PROPOSITION X.

Problem.

FIG. 27. 63. *TWO Conjugate Diameters* A a, B b, *of an Ellipsis being given, as also a right Line* MCm, *in passing thro' the Centre* C, *to find the Points* M, m *in that Line, wherein it meets the Ellipsis.*

Draw

Draw the indefinite Line AD thro' A, one End of the Diameter Aa, parallel to Bb the Conjugate Diameter, meeting the Line CM given in Position in the Point D; moreover, draw the Line AO thro' the Point A, perpendicular to AD, and equal to CB, and the Line OD thro' the Points O, D. This being done, about the Centre O, with the Radius OA, describe a Circle OA, cutting the Line OD in the two Points N, n; and then if NM, nm be drawn from these Points parallel to the Line OC, which joins the Centres of the Circle and Ellipsis; I say the Points M, m, wherein they meet the Line CD, will be in the Ellipsis, and consequently will determine the Extremities of the Diameter MCm given in Position.

For draw the Lines MP, NQ, parallel to AD meeting the Line CA, OA in the Points P, Q; then the similar Triangles CDO, MDN, and CDA, CMP, and ODA, ONQ will give these Proportions $CA : CP :: CD : CM :: OD : ON :: OA : OQ$. That is, $CA : CP :: OA : OQ$. And therefore if the right Line PQ, be drawn, it will be parallel to OC; and consequently also to MN which is supposed parallel to OC. So the Parallels MP, NQ, will be equal to each other. This being supposed, if we call the given Lines CA, t; CB or AO or ON, c; and the undeterminate ones CP, x; PM or NQ, y; we shall have this Proportion $CA\,(t) : CP\,(x) :: OA\,(c) : OQ = \frac{cx}{t}$. And because the Triangle OQN is Right-angl'd at Q, the Square $\overline{NQ}^2$ or $\overline{MP}^2\,(yy) = \overline{ON}^2\,(cc) - \overline{OQ}^2\left(\frac{ccxx}{tt}\right)$. Whence the Line MP will be * an Ordinate to the Diameter Aa, and consequently the Point M will be in the Ellipsis, whose Conjugate Diameters are Aa, Bb. But because the Lines NM, OC, nm, are parallel, the Line Mm is bisected by the Centre C; since (by the Property of the Circle) Nn is bisected in O. Therefore the Point m will be * likewise in the same Ellipsis.

* Art. 41, and 55.

* Art. 50.

If the two Conjugate Diameters Aa, Bb, should happen to be the Axes, then the Parallels CO, PQ, would Coincide with the Lines CA, AO, all four of which would make but one streight Line. And by this Means the Construction and Demonstration would have been something easier.

PROPOSITION XI.

Problem.

64. TWO *Conjugate Diameters* Aa, Bb, *of an Ellipsis, being given; to find the Axes* (Mm, Ss) *thereof: And demonstrate that an Ellipsis can have but two Axes.* Fig. 27.

Draw

Draw the Line DE through A one End of the Diameter Aa, parallel to Bb the Conjugate Diameter, and the Line AO perpendicular to DE, and equal to CB. Then join OC, and draw the Line FG thro' F the middle Point thereof perpendicular to the same, and meeting the Line DE in the Point G, on both sides of which Point take the equal Parts GD, GE, in the Line DE, each equal to GO or GC. This being done, if the right Lines CD, CE be drawn: I say the two Axes Mm, Ss, are situate in these Lines.

† Art. 58. For since the two Axes may be esteem'd † as Conjugate Diameters, cutting one another at right Angles, they will meet the Line DE in the Points D, E, such, that a Circle describ'd on that Line as a Diameter, will pass through the Points C, O; because the Rectangle

* Art. 62. $DA \times AE$ being equal * to the Square of AO, the Angle DOE will be a right one, as well as the Angle DCE. But it is manifest, that this is entirely what the foregoing Construction has effected; since the Lines GO, GC, GE, GD, being all equal to one another, are so many Radii of the same Circle. But since there are no other Points in the Line DE, but D, E, which at the same time can satisfy these two Conditions, *viz.* that the Angles DCE, DOE be each right Angles; therefore the two Conjugate Diameters Mm, Ss, which are at right Angles to each other, will be the Axes, and there are only two of them.

Now to determine the Lengths of the Axes, you need only draw the right Lines OD, OE, meeting the Circle whose Radius is OA, in the Points N, R, and then the Parallels NM, RS. For it is evident,

* Art. 63. * that M, S, the Points wherein these Parallels meet the right Lines CD, CE, appertain to the Ellipsis, whose Conjugate Diameters are the Lines Aa, Bb; and so the Points M, S, will be Extremities of the two Axes.

COROLLARY.

65. IF it had been requir'd to find two Conjugate Diameters Mm, Ss, that might cut each other in the Angle MCS, equal to a given Angle; two other Conjugate Diameters Aa, Bb, being given: It is plain that the Problem might have been reduced to this, *viz.* to find two Points D, E, in the Line DE given in position such; that if the right Lines DO, OE, CD, CE, are drawn to the two Points O, C, given without the Line DE, the Angle DOE may be a right Angle, and the Angle DCE, equal to a given Angle. But since the Solution of this Problem is pretty difficult, I refer it to the 10th Book, and here follow another Manner, which is more simple; and that is, to find first the two Axes, and then by means of them, the two Conjugate Diameters sought, as we are going to shew in the following Problem.

PRO-

PROPOSITION XII.

Problem.

66. THE *two Axes* A a, B b *of an Ellipsis being given, to find two Conjugate Diameters* M m, S s, *cutting one another in the Angle* M C S, *equal to an Angle given.* FIG. 28. and 29.

Let us suppose the Diameters Mm, Ss, to be those requir'd, and that they meet the indefinite right Line DE (drawn thro' A, the End of the little Axis Aa, parallel to the great Axis Bb) in the Points D and E. Now draw the Line CF from C, the Centre of the Ellipsis, making the Angle CFE, at the Point F, with the Line DE, equal to the given Angle MCS, and call the given Lines CA, t; CB, c; AF, a; and the unknown one AE, z; then will * $AD = \frac{cc}{z}$, and $CE = \sqrt{tt + zz}$, because CAE is a right-angled Triangle. This being suppos'd, * Art. 62.

The Triangles FEC, CED will be similar; because the Angle at the Point E is common, and the Angle CFE was made equal to the Angle MCS; therefore $FE\ (z-a) : EC\ (\sqrt{tt+zz}) :: EC\ (\sqrt{tt+zz}) : ED\ (z + \frac{cc}{z})$. Whence by multiplying the Means and Extremes, this Equation will be formed $zz - az + cc - \frac{acc}{z} = tt + zz$, and striking out zz from both Sides, multiplying by z, and dividing by a, there will come out $zz - \frac{cc}{a}z + \frac{tt}{a}z + cc = 0$. And (for Brevity's Sake) putting $\frac{cc - tt}{a} = 2b$; the last Equation will become this, $zz - 2bz + cc = 0$, or $zz - 2bz + bb = bb - cc$. And by extracting the square Roots of both Sides there comes out $z - b$, or $b - z = \sqrt{bb - cc}$; and consequently the unknown Quantity $AE\ (z) = b \pm \sqrt{bb - cc}$, which last Equation gives the following Construction.

Produce the small Axis Aa to the Point O, so that AO be equal to CB, the half of the great Axis, and draw CF, making the Angle CFE, with the Line DE drawn through A, parallel to Bb, equal to the given Angle; join OF, and draw the right Lines OH, CG, perpendicular to OF, CF, meeting the Line DE in the Points H, G (the Points H, G, in the 28th and 29th Figures, are not denoted in the Line DE; because doing that would have enlarged the Figures too much,

much, and since it is easy to imagine them) This being done, about the Centre O, with the Radius OK, equal to the half of GH, (that Part of AD produced, which is comprehended between G,H,) describe an Arc of a Circle cutting DE in the Points K,K; then if KD, KE, be taken in DE, each equal to KO, and the right Lines DC, EC, are drawn through C the Centre of the Ellipsis; I say, the Diameters sought Mm, Ss, are situate in DC, EC.

For because FAC, FCG, and FAO, FOH, are right Angles, we shall have $AG = \frac{tt}{a}$, $AH = \frac{cc}{a}$; and therefore $GH = \frac{cc-tt}{a} = 2b$. Whence the Radius OK, which is equal to $\frac{1}{2}GH$, will be equal to b. And because OAK is a right-angled Triangle, we have $AK = \sqrt{bb-cc}$, and AE or $KE \mp AK = b \mp \sqrt{bb-cc}$, and AD or $KD \pm AK = b \pm \sqrt{bb-cc}$. Now this being suppos'd, if the Value of AE be multiplied by that of AD, there will arise $AE \times AD = cc = \overline{CB}^2$; and therefore * Mm, Ss are Conjugate Diameters. But the Rectangle under $AE + AD$ or DE $(2b)$ and $AE - AF$ or EF $(b \mp \sqrt{bb-cc} - a)$ is $= 2bb \mp 2b\sqrt{bb-cc} + 2ab = 2bb \pm 2b\sqrt{bb-cc} + tt - cc$ by putting $cc-tt$ for $2ab$ the Value thereof; and because the Triangle CAF is right-angled, the Square $\overline{CE}^2 = \overline{AE}^2 + \overline{CA}^2 = 2bb \mp 2b\sqrt{bb-cc} + tt - cc = DE \times EF$; and so $FE : EC :: EC : ED$. Therefore the Triangles FEC, CED will be similar; because the Angle at the Point E is common, and the Sides about that Angle are proportional. Whence the Angle MCS will be equal to the given Angle CFE. Which is what was to be demonstrated.

* Art. 62.

Now to determine the Lengths of CM, CS, the two Semi-Diameters sought, you need only draw the Lines OD, OE, and then the Lines NM, RS, parallel to OC, thro' the Points N, R, wherein OD, OE, meet the Circle, whose Radius is OA. For it is manifest, * that the Points M, S, wherein NM, MS meet the Lines CD, CE, will be in the Ellipsis, and consequently do determine the Extremities of the Diameters.

* Art. 63.

COROLLARY I.

67. IT follows from the foregoing Construction, 1. That when the Problem is possible, OK $\left(\frac{cc-tt}{2a}\right)$ must exceed or be equal to AO (c); for otherwise the Circle describ'd with the Radius OK, will not meet the Line DE, and so the Problem will in this Case be impossible.

2. When

2. When OK exceeds OA, we can always find two different Pair of Conjugate Diameters Mm, Ss, by means of the Points K, K, that will answer the Question: But then the Diameter Ss, of *Fig.* 29. is equal to the Diameter Mm of *Fig.* 28. and alike posited on the other Side of the Axis Aa; because AE of *Fig.* 29. is equal to AD of *Fig.* 28. and moreover, the Diameter Mm of *Fig.* 29. is equal to the Diameter Ss of *Fig.* 28. and alike situate on the other Side of the Axis Aa; because AD of *Fig.* 29. is equal to AE of *Fig.* 28. that is, the two different Pair of Conjugate Diameters Mm, Ss, which equally answer the Problem, are alike situate on each Side of the Axis Aa; and their Magnitudes will remain the same in those two different Positions.

3. When $OK = OA$, the two Points of Intersection K, K will coincide in the Point of Contact A; and then you need but take AE, AD, each equal to CB, the half of the great Axis. Whence it appears, that in this Case the Problem is capable but of one Solution; and the two Conjugate Diameters Mm, Ss, which solve it, are equal between themselves. Fig. 30.

Corollary II.

68. It is manifest also, that the greater AF (a) is, the greater is the given obtuse Angle CFE, and contrariwise the less will the Line OK $\left(\frac{cc-tt}{2a}\right)$ be: So that when AF is the greatest possible, the obtuse Angle CFE will be also the greatest possible; and contrariwise, the Line OK will be the least possible, *viz.* equal to AO. But then, if the right Lines Ba, ab, are drawn, the right-angled Triangles aCB, CAD, aCb, CAE will be all equal to one another; because the Lines AE, AD are each equal to CB or Cb, the Halfs of the Axis Bb, and CA is equal to Ca. And therefore the Angle ACM will be equal to the Angle CaB, and $ASC = Cab$; therefore the given Angle MCS or CFE will be equal also to the Angle Bab. Hence it follows, Fig. 28, 29, and 30. Fig. 30.

1. If the Lines aB, ab are drawn from a, one End of the little Axis Aa to B, b, the Ends of the great one; the given obtuse Angle CFE must be equal or less than the Angle Bab, that so* the Problem may be possible. Fig. 28, 29, and 30. * Art. 67.

2. When the given obtuse Angle CFE is $= Bab$, as in *Fig.* 30. then there are only two Conjugate Diameters Mm, Ss, answering the Problem; and they are equal to one another.

3. And when the Angle CFE is less than Bab, as in the 28th and 29th Figures; then there will be always two different Pair of Conjugate Diameters answering the Problem, being situate similarly on both

 Sides

Sides the little Axis, the said Angle CFE between them remaining the same, and their Magnitude will be equal.

PROPOSITION XIII.

Problem.

69. *TWO Conjugate Diameters* A a, B b, *of an Ellipsis, being given; to describe the same by a continued Motion.*

* Art. 64. First find * the two Axes, and then describe the Ellipsis by the Directions in Art. 36.

Fig. 31, 32. But this may be done another way, which is thus. Draw the right Line AH through A one End of the given Diameter Aa, perpendicular to the other Diameter Bb, and take AQ in the said Line, on either side the Point A, equal to CB, and draw the Line CQ; then if the Line GF, equal to HQ, be mov'd so, that the Ends thereof be always in the Lines Bb, CQ, (produced both ways from the Centre C, as is necessary) till it has mov'd successively through the four Angles made by the two Lines, and is come again to the same Situation from which it went; I say, if GM be taken equal to AQ, the Point M by this Motion will describe the Ellipsis sought.

For draw GP parallel to QA, meeting the Diameter Aa in P, and the Diameter Bb in O; then the similar Triangles CHQ, COG, and CAQ, CPG, will give this proportion, $CQ : CG :: AQ$, or $GM : GP :: HQ$, or $GF : GO$. And so the Line PM will be parallel to the Diameter Bb. This being supposed,

Call the given Lines CA, t; AQ, or CB, c; and the unknown ones CP, x; PM, y; then we shall have $CA\,(t) : CP\,(x) :: AQ\,(c) : GP = \frac{cx}{t}$. And the right-angl'd Triangle GPM will give $\overline{PM}^2 =$

* Art. 41, 55. $\overline{GM}^2 - \overline{GP}^2$, that is $yy = cc - \frac{ccxx}{tt}$. Whence PM will be * an Ordinate to the Diameter Aa in the Ellipsis, whose Conjugate Diameters are the Lines Aa, Bb. Therefore, &c.

If the two Conjugate Diameters Aa, Bb, were the Axes; then it is manifest that the Lines AQ, CQ, would fall in the Diameter Aa, which would be one of the Axes, and the Point H would fall in the Centre C. Whence it appears, that in this Case, GF must have been then taken equal to CQ, the Sum or Difference of the two Semi-Axes CA, CB; and the Ends thereof mov'd along the Axes Aa, Bb produced, as is necessary.

Fig. 33.

Because

Because the right Lines $A a$, $B b$, cut one another at right Angles in the Point C, it is manifest, that in whatsoever situation the right Line $G F$ is found, during the Motion of its Ends along these Lines, the Circle that should have that Line for a Diameter, would always pass through the Point C: And so the Line $C D$ passing through D the middle of the Line $F G$, will be always equal to $D F$, because the Lines $C D$, $D F$, $D G$, will always be Radii of that Circle. Whence arises the following Description.

Let the right Lines $C D$, $D F$, be each equal to the half of $C Q$, the Sum or Difference of the two Semi-Axes $C B$, $C A$, and fasten them so together at their common End D, that they may move about the same, like the two Legs of a pair of Compasses about the Head. This being done, fasten C the Extremity of the Line $C D$ in the Centre of the Ellipsis, and move F the End of the other Line $F D$, along the Axis $B b$, so that it causes the Side $C D$ to move about the fixed Point C. Then it is evident that the Point M taken in $F D$ (produced, if necessary) so that $F M$ be equal to $C A$, will by this Motion describe the Ellipsis sought.

PROPOSITION XIV.

Theorem.

70. *TWO Conjugate Diameters* (A a, B b,) *of an Ellipsis being given; to describe the same through several Points.*

Draw the indefinite right Line $D A D$, through A one End of the given Diameter $A a$, parallel to $B b$ the Conjugate Diameter, and draw $A O$ perpendicular to $A D$, and equal to ($C B$) half the Diameter $B b$, and join $O C$, and about the Centre O with the Radius $O A$, describe a Circle. This being done on both Sides of $C A$, draw at pleasure any Number of Lines $C D$, $C D$, &c. from the Centre C, and then draw the Lines $O D$, $O D$, &c. from the Centre O to the Points D, D, &c. cutting the Arc of the Circle in the Points N, N, &c. and draw the right Lines $N M$, $N M$, &c. parallel to $C O$, and meeting the correspondent right Lines $C D$, $C D$, &c. in the Points M, M, &c. then if the Points m, m, &c. are mark'd in the right Lines $C M$, $C M$, &c. continued, equally distant from C; it is manifest, * that the Curve Line passing through all the Points M, M, &c. m, m, &c. thus found, will have the right Lines $A a$, $B b$, for two Conjugate Diameters. FIG. 34. * Art. 63.

This may be done otherwise thus; divide $C B$, one of the Semidiameters into as many equal Parts, $C E$, $E E$, &c. as possible, and draw the

the Perpendiculars ED, ED, &c. meeting the Arc of a Circle describ'd about the Centre C, with the Radius CB, in the Points D, D, &c. Join AB, and draw the Line EP through E, one of the aforesaid Points (that is nighest to the Centre C,) parallel to AB, meeting CA in P. Then if in the Diameter Aa, be taken the equal Parts PP, PP, &c. on both Sides of the Centre C, each equal to CP, and through the Points P, P, &c. be drawn the Lines PM, PM, &c. parallel to the Diameter Bb, (on both Sides of A) each equal to its Correspondent ED: I say the Curve Line passing through all the Points M, M, &c. will be in the Ellipsis sought.

For call the given Quantities CA, t; CB or CD, c; and the indeterminate one CP, x; PM, y; then because the Triangles CAB, CPE, are similar, we have this Proportion, $CA\,(t) : CB\,(c) :: CP\,(x) : CE = \frac{cx}{t}$. And because the Triangle CED is Right angl'd at E, the Square $\overline{ED}^2$ or $\overline{PM}^2\,(yy) = \overline{CD}^2\,(cc) - \overline{CE}^2\,\left(\frac{ccxx}{tt}\right)$. Therefore the

* Art. 41, 55. Line PM will * be an Ordinate to the Diameter Aa; and since this Demonstration extends to all the right Lines PM; because every CP is to its Correspondent CE, in the Ratio of CA to CB; therefore the Curve passing through all the Points M, M, &c. found as above, will be the Ellipsis sought.

The End of the Second Book.

BOOK III.

Of the Hyperbola.

DEFINITIONS.

I.

IF one End of a long Rule fMO be fasten'd in the Point f, taken on a Plane, in such a manner, that it may turn freely about that fix'd Point f, as a Centre; and if one End of the Thread FMO, (being in Length less than the said Rule) be fixed to O, the other End of the Rule, and the other End of the Thread be fix'd in the Point F taken on the Plane. Then if the Rule fMO be turn'd about the fix'd Point f; and at the same time you keep the Thread OMF always in an equal Tension, and its Part MO close to the Side of the Rule, by means of the Pin M: The Curve Line AX describ'd by the Motion of the Pin M, is one Part of an *Hyperbola*. FIG. 36.

And if the Rule be turn'd about, and move on the other Side of the fixed Point F, the other Part AZ of the same Hyperbola may be describ'd after the same manner.

But if the End of the Rule be fasten'd in F, and that of the Thread in f, (the Rule and Thread keeping the same Lengths) you may describe another Curve Line xaz after the said manner, which will be opposite to XAZ, and is called likewise an Hyperbola; and both these two Curves together are called *opposite Hyperbola's*, [or opposite Sections.]

2.

The two fixed Points F, f, are called the *Foci*.

3.

The Line Aa, which passes thro' the two Foci F, f, and terminating both ways in the opposite Hyperbola's, is called the *first Axis* [or *principal Axis*.]

4.

The Point C, dividing the first Axis Aa in the middle, is called *the Centre*.

4 5. If

5.

If the indefinite right Line Bb, be drawn through the Centre C, perpendicular to the first Axis Aa; and if about the Point A, as a Centre, with the Distance CF, an Arc of a Circle be describ'd cutting Bb, in the Points B, b: Then the Part Bb of that perpendicular, is called the *Second Axis*, [or the *Conjugate Axis.*]

6.

The two Axes Aa, Bb, are together call'd *Conjugate Axes*; so that the first Axis Aa is said to be a Conjugate to the Second Bb; and contrariwise the Second Bb, a Conjugate to the first Aa.

7.

The Lines MP, MK, drawn from Points (M) of the opposite Hyperbola's parallel to one of the Conjugate Axes, and terminating in the other, are call'd *Ordinates* to that Axis: So MP is an Ordinate to the first Axis Aa, and MK one to the second Bb.

8.

A third Proportional to the two Axes, is call'd the *Parameter* of that which is the first Term of the Proportion. So if you make as the first Axis Aa is to the second Bb, so the second Bb to a third Proportional p; then the Line p will be the Parameter of the first Axis Aa.

9.

All Lines passing through the Centre C, are called *Diameters*: Those which meet the opposite Hyperbola's, being *First Diameters*, [or principal Diameters] and those which being infinitely produced, do not meet them, *Second Diameters*, [or *Conjugate Diameters.*]

10.

A right Line which meets an Hyperbola but in one Point, and being both ways continued, falls without the Hyperbola, is called a *Tangent* to it in that Point.

SCHOLIUM.

FIG. 37. 71. THE reason why we have said in the first Definition, that the Length of the Thread FMO must be less or greater than the Length of the Rule fMO, is, because if the Thread was equal to it, the Pin M by its motion, would describe a Line, having all the Points (M) thereof equally distant from the two Points F, f; because if MO, the common Part of the Thread, were taken from the Thread and Rule, the Parts MF, Mf, remaining, would always be equal between

between themselves. Whence it is manifest, that the Line describ'd by the Pin M would, in that Case, be the indefinite right Line Bb, drawn through C the Middle of Ff, perpendicular to Ff.

COROLLARY I.

72. IT follows from the first Definition, that if the right Lines MF, Mf, be drawn from any Point (M) of one of the opposite Hyperbola's to the two Foci F, f; their Difference $MF-Mf$ will be always the same: for it will be always equal to the Difference between the Lengths of the Rule and Thread. FIG. 36.

COROLLARY II.

73. WHEN the Point M falls in A, it is evident, that MF will become Af; and further, when the Point M falls in a, in describing the opposite Hyperbola xaz; it is manifest, that MF will become aF, and Mf will become af. Whence because the Difference ($MF-Mf$) of MF and Mf, is always the same, we shall have $Af-AF$, or $Ff-2AF=aF-af$, or $Ff-2af$: and therefore $AF=af$. Whence it follows,

1. That the focal Distance Ff is divided into two equal Parts by the Centre C: because $CA+AF$, or $CF=Ca+af$ or Cf.

2. That the Difference of the two right Lines MF, Mf, is always equal to the first Axis Aa; because, in the Hyperbola XAZ, we have always $Mf-MF=Af-AF$, or $Af-af$; and in the opposite Hyperbola we have likewise always $MF-Mf=aF-af$, or $aF-AF$.

COROLLARY III.

74. IT follows from the fifth Definition,

1. That the second Axis Bb is divided into two equal Parts by the Centre C; for the right-angled Triangles ACB, ACb, will be equal; because the Hypothenuses AB, Ab are equal, and the Side AC is common.

2. If CE be taken in the second Axis Bb, equal to CA the half of the first Axis, and the Hypothenuse AE be drawn; then the second Axis Bb will be greater, equal to, or less than the first Aa; according as the right Line CF is greater, equal to, or less than the Hypothenuse AE; because the Hypothenuse Ab being taken equal to CF, will then be found likewise greater, equal to, or less than the Hypothenuse AE.

3. If CF, Cf, be taken in the first Axis Aa, (on both Sides the Centre C) each equal to AB, the Hypothenuse of the right-angled Triangle CAB, formed by the two Semi-axes CA, CB: then the Points F, f, will be the two Foci.

COROLLARY IV.

75. THE same Things being premised, if you call the given Quantities CF or AB, m; CA or Ca, t; then the right-angled Triangle ACB, will give $\overline{BC}^2 = mm - tt$. But $AF = m - t$, and $Fa = m + t$; and therefore $AF \times Fa = mm - tt$. Whence it is manifest, that the Square of CB, the half of the second Axis Bb, is equal to the Rectangle under AF, and Fa, the Parts of the first Axis Aa, comprehended between one of the Foci F, and A, a, the two Ends of that Axis.

COROLLARY V.

76. IT will not be difficult now to describe the opposite Hyperbola's, having the two Axes Aa, Bb given, supposing it to be known that Aa is the first Axis. For if the two Foci F, f, be * found in the first Axis Aa, and one End of a Thread FMO be fixed in the Point F; and if then you fix O, the other End of that Thread, to the End of a long Rule OMf, (whose Length must * be less or greater than the Length of the Thread OMF, by the Length of the Line Aa.) And if the other End of that Rule be fasten'd in the Focus f, so as to move about the same: Then may you describe the two opposite Hyperbolas XAZ, xaz, as is directed in *Def.* 1. and it is evident, that the Line Aa will be their first Axis, and the Line Bb the second.

* Art. 74.

* Art. 71.

Note, The longer the Rule OMf is, the greater will the Parts of the opposite Hyperbola's describ'd, by means thereof, be; so that they may be augmented at pleasure, by equally augmenting the Length of the Rule and Thread.

PROPOSITION I.

Theorem.

77. *IF an Ordinate* MP *be drawn to the first Axis* Aa, *and* AD *be taken in that Axis (produced) equal to* MF, *from* A *towards the Focus* F, *when the Point* M *is in the Hyperbola* XAZ, *and towards the Focus* f, *when that Point does fall in the opposite Hyperbola* xaz. *I say,* CA : CF :: CP : CD.

Call

Call (as before) the given Quantities CA or Ca, t; CF, or Cf, m; and moreover, the indeterminate Quantities CP, x; PM, y; and the unknown Quantity CD, z; in the first Case, we shall have AD or $MF = z - t$, aD or $Mf = z + t$, $FP = x - m$, or $m - x$ (according as the Point P falls below or above the Focus F), $Pf = x + m$: And in the second Case, AD or $MF = z + t$, aD or $Mf = z - t$, $FP = x + m$, $Pf = x - m$ or $m - x$, according as the Point P falls above or below the Focus f.

Now the right-angled Triangle MPF will give $zz \mp 2tz + tt = yy + xx \mp 2mx + mm$; *viz.* $-$ in the first, and $+$ in the second Case; and the other right-angled Triangle MPF will give $zz \pm 2tz + tt = yy + xx \pm 2mx + mm$; *viz.* $+$ in the first, and $-$ in the second Case.

Then if each Member of the first Equation, in the first Case, be orderly taken from those of the second Equation; and contrariwise, (in the second Case) each Member of the second Equation from those of the first, there will be had $4tz = 4mx$: Whence CD (z) will be $= \frac{mx}{t}$. Therefore $CA\ (t) : CF\ (m) :: CP\ (x) : CD\ (z)$. *W. W. D.*

COROLLARY.

78. HENCE if you call the given Quantities CA or Ca, t; CF, or Cf, m; and the indeterminate Quantity CP, x; it is evident, we shall have always $MF = \frac{mx}{t} - t$, and $Mf = \frac{mx}{t} + t$, when the Point M happens in the Hyperbola XAZ, whose Focus is the Point F: and contrariwise, $MF = \frac{mx}{t} + t$, and $Mf = \frac{mx}{t} - t$, when the Point M falls in the opposite Hyperbola xaz, whose Focus is the Point f.

PROPOSITION II.

Theorem.

79. *THE Square of any Ordinate* (PM) *to the first Axis* Aa, *is to the Rectangle under* AP, Pa, *the Parts of the Axis produced, as the Square of the Conjugate Axis* Bb, *to the Square of the first Axis* Aa.

We are to prove, that $\overline{PM}^2 : AP \times Pa :: \overline{Bb}^2 : \overline{Aa}^2$.

The same Things being premised as in the last Proposition, if $\frac{mx}{t}$ be put for its Value z, in the Equation $zz \mp 2tz + tt = yy + xx \mp 2mx + mm$, found

* *Art.* 77. found by means * of the right-angled Triangle MPF, we shall have this Equation always formed, *viz.* $ttyy = mmxx - mmtt - ttxx + t^4$, which being reduced to a Proportion, and then $\overline{PM}^2 (yy) : AP \times Pa\ (xx - tt)$

* *Art.* 75. $:: \overline{BC}^2 * (mm - tt) : \overline{CA}^2 (tt) :: \overline{Bb}^2 : \overline{Aa}^2$. *W. W. D.*

COROLLARY I.

80. IF an Ordinate (MK) be drawn to the second Axis Bb, which call $2c$; then it is manifest, that $MK = CP\,(x)$, and $CK = PM\,(y)$. But $\overline{PM}^2 (yy) : AP \times Pa\ (xx - tt) :: \overline{Bb}^2 (4cc) : \overline{Aa}^2 (4tt)$. And therefore $4ccxx = 4cctt + 4ttyy$; from whence we get this Proportion, $\overline{MK}^2 (xx) : \overline{CK}^2 + \overline{CB}^2 (yy + cc) :: \overline{Aa}^2 (4tt) : \overline{Bb}^2 (4cc)$.

That is the Square of any Ordinate (MK) to the second Axis Bb, is to the Square of CK *plus* the Square of CB, the half of the second Axis, as the Square of the Conjugate Axis Aa, to the Square of the second Axis Bb.

A Fundamental COROLLARY. I.

FIG. 38, and 39. 81. IF the first Axis Aa be called $2t$; the second Axis Bb, $2c$; the Parameter p; each of the Ordinates PM, y; and each of the correspondent Parts (CP), contain'd between the Centre, and the

* *Art.* 79, and 80. Points of Concurrence of the Ordinates, x; we shall have * always $\overline{PM}^2 (yy) : \overline{CP}^2 \mp \overline{CA}^2 (xx \mp tt) :: \overline{Bb}^2 (4cc) : \overline{Aa}^2 (4tt) :: p : Aa\ (2t)$. Because, by the Definition of a Parameter, $Aa\ (2t) : Bb\ (2c) :: Bb\ (2c) : p = \frac{4cc}{2t}$. Where you must observe, that it is $xx - tt$, when Aa is the first Axis, and then the Rectangle $AP \times Pa$ may be substituted for $\overline{CP}^2 - \overline{CA}^2$; and on the contrary, it is $xx + tt$, when Aa is the second Axis. Hence multiplying the Extremes, and Means of the first Proportion $yy : xx \mp tt : 4cc : 4tt$. And then those of the other $yy : xx \mp tt : p : 2t$. there will arise $yy = \frac{ccxx}{tt} \mp cc$, and $yy = \frac{pxx}{2t} \mp \frac{1}{2} pt$. And since this Property agrees equally to all the Points of the opposite Hyperbola's, and determines their Position with regard to the Axes; therefore the Equation $yy = \frac{ccxx}{tt} \mp cc$, or $yy = \frac{pxx}{2t} \mp \frac{1}{2} pt$, entirely expresses the Nature of the Hyperbola with regard to the Axes.

COROLLARY III.

82. IF any two Ordinates (MP, NQ) be drawn to the Axis Aa, it is manifest that $\overline{MP}^2 : \overline{QN}^2 :: \overline{CP}^2 \mp \overline{CA}^2 : \overline{CQ}^2 \mp \overline{CA}^2$. For $\overline{PM}^2 : \overline{CP}^2 + \overline{CA}^2 :: \overline{Bb}^2 : \overline{Aa}^2 :: \overline{QN}^2 : \overline{CQ}^2 + \overline{CA}^2$. Whence, &c.

It is necessary here to take notice, that the Rectangles $AP \times Pa$, $AQ \times Qa$ may be substituted for $\overline{CP}^2 - \overline{CA}^2$ and $\overline{CQ}^2 - \overline{CA}^2$ as being equal to them; which I would have hereafter always observed.

COROLLARY IV.

83. IF any right Line MPM be drawn through any Point (P) of either Axis, as Aa (produc'd if it be the first Axis) parallel to Bb the Conjugate Axis to Aa; then that Line will meet one or both the opposite Hyperbola's in only two Points M, M, equally distant from the Point P. For in order that the Points M, M, be in one or both the Hyperbola's, it is necessary * that the Squares of both the PM^s (y) on each Side the Axis Aa, be each equal to the same Quantity $\frac{ccxx}{tt} \mp cc$. * Art. 81.

COROLLARY V.

84. HENCE it follows, that since yy is $= \frac{ccxx}{tt} \mp cc$, the more CP (x) (taken on either Side the Centre C) increases, the more will the correspondent Ordinates ($PM = y$), on each Side of the Axis (Aa) likewise increase, even infinitely: And contrariwise, the more CP (x) dimishes, the more will PM (y) likewise diminish; so that (*Fig.* 38.) CP (x) being equal to CA or Ca (t) when Aa is the first Axis; PM (y) will then be equal to nothing, and (*Fig.* 39.) CP (x) being equal to nothing, when Aa is the second Axis, both the PM's, (y) which then will become CB or Cb (c), are less than any of the Ordinates ($PM = y$) taken on both Sides the Centre. Whence it is manifest; FIG. 38, and 39.

1. If Parallels be drawn (*Fig.* 39.) to the second Axis Aa, thro' B, b, the Ends of the first Axis Bb; then these Parallels will touch the opposite Hyperbola's in the Points B, b.

2. The opposite Hyperbola's recede more and more from their Conjugate Axes, even infinitely, beginning from the Extremes of the first Axis; but yet with this Difference, that the first Axis meets each of the opposite Hyperbola's in one Point, and being continued, will be

be ever after within them; whereas the second Axis falls quite without the Hyperbola's, and being infinitely produced, will never meet either of them.

COROLLARY VI.

85. IT follows, because yy is $= \frac{ccxx}{tt} \mp cc$, that if the Points P, P, be taken on both Sides the Centre (C) equally distant from the same; then the Ordinates PM, PM, will be equal. Whence it is manifest, that if a right Line MM, terminating in one Hyperbola, or in the opposite ones, be cut into two equal Parts by an Axis Bb, in the Point K not being the Centre, that Line will be parallel to the Conjugate Diameter Aa. For if MP, MP be drawn parallel to the Axis Bb, then the Line PP, will be bisected in C, because MM is so in K; and therefore the Ordinates PM, PM, will be equal, and the right Line MM parallel to the Axis Aa.

COROLLARY VII.

* Art. 8 86. IF the two Parts of the Plane, whereon the opposite Hyperbola's are drawn, on each Side of the Axis Aa, be conceiv'd to be
FIG. 39. folded together, it is manifest, that when Aa is the second Axis, the two opposite Hyperbola's will exactly agree, or coincide, *viz.* the Points B, M, &c. with the Points b, M, &c. because all the Perpen-
* Art. 83. diculars Bb, MM, &c. drawn to that Axis, are * bisected in the Points C, K, &c.

And for the same Reason (*Fig.* 38.) when Aa is the first Axis, the Parts of the opposite Hyperbola's on each Side the Axis will perfectly agree or coincide.

ADVERTISEMENT.

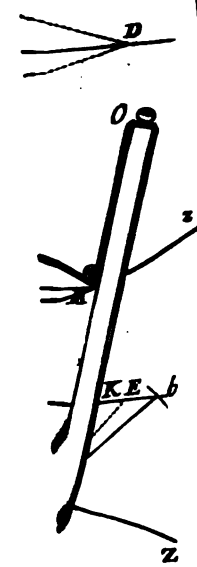
We have hitherto, in this Book, kept to the same Method, as in the Ellipsis, and might have continued it on to the End; but because the other Properties of the Hyperbola may be easier demonstrated from certain particular Lines appertaining to the same, which must necessarily be spoken of; therefore I here deviate from that Method, and first lay down the following Definitions of these Lines, and afterwards demonstrate the remaining Properties by means of them.

DEFINITIONS.

II.

FIG. 40. If from the Centre C the indefinite right Lines CG, Cg, be drawn parallel to the Lines Ab, AB, drawn from the End A of the first Axis

35



Axis Aa, to the two Ends B, b, of the second; then the said right Lines CG, Cg, are called the *Asymptotes* of the Hyperbola MAM; and being infinitely produced on the other Side of the Centre, they are called the *Asymptotes* of the opposite Hyperbola MaM.

12.

The Square of CG or Cg, that Part of an Asymptote contain'd between the Centre C, and the Point of Concurrence of the Line AB, or Ab, drawn from A, the End of the first Axis, to B, or b, the Extremity of the second Axis, is called the *Power* of the Hyperbola MAM, or MaM.

COROLLARY I.

87. HENCE it is evident, that the Angle GCg (or its equal BAb) form'd by the Asymptotes of an Hyperbola, is lesser, equal to, or greater than a right Angle; according as the second Axis Bb is less, equal to, or greater than the first Aa. For when the first Axis Aa exceeds the second Bb; then (CA) the half thereof, will exceed CB, the half of the second Axis: and consequently CAB, the Angle of the right-angled Triangle CAB, is less than half a right Angle; therefore the two equal Angles CAB, CAb, which together make up the Angle BAb, will be less than a right Angle. And after the same manner may the two other Cases be demonstrated.

COROLLARY II.

88. BEcause the Triangles BAb, BGC, are similar, it is plain that the Line AB is divided by the Asymptote CG into two equal Parts, in the Point G, and also CG is equal to half of Ab, because BC is the half of Bb. After the same manner we prove, that Ab is divided by the Asymptote Cg, in two equal Parts in the Point g, and Cg is the half of AB. Therefore all the Lines CG, GA, GB, Cg, gA, gb, are equal to each other: Because every of them is equal to the half of AB, or Ab, which by *Def.* 5. are equal to one another.

COROLLARY III.

89. THE Power of an Hyperbola is equal to one fourth Part of the Sum of the Squares of the two Semi-axes. For if you call CA, t; CB, c; CG, m; we shall have * $BA = 2m$. And because ACB is a right-angled Triangle, the Square $\overline{AB}^2$ ($4mm$) will be $= tt + cc$. And consequently $\overline{CG}^2$ (mm) $= \frac{tt+cc}{4}$.

* *Art.* 88.

PRO-

I

PROPOSITION III.

Theorem.

FIG. 40. 90. *IF from any Point* M *in either of the opposite Hyperbola's, be drawn a a right Line* R r *perpendicular to the first Axis* A a, *(meeting the same in* P,) *and terminating in the Asymptotes in* R *and* r: *I say, the Rectangle under* RM, Mr, *is equal to the Square of* BC, *the half of the second Axis* B b.

We are to prove, that $RM \times Mr = \overline{BC}^2$.

Call the known Quantities CA, t; CB, c; and the indeterminate ones, CP, x; PM, y; then because the Triangles ACB, CPr, and ACb, CPR, are similar; therefore $CA\ (t) : CB$ or $Cb\ (c) :: CP\ (x) :$ Pr, or $PR = \frac{cx}{t}$. Whence RM or $PR \pm PM = \frac{cx}{t} \pm y$; and Mr, or $Pr \mp PM = \frac{cx}{t} \mp y$. And consequently $RM \times Mr = \frac{ccxx}{tt}$

* Art. 81. $- yy = \overline{BC}^2\ (cc)$ in substituting yy for its Value * $\frac{ccxx}{tt} - cc$. *W.W.D.*

COROLLARY. I.

91. IT is manifest, that $\overline{PM}^2 \left(\frac{ccxx}{tt} - cc\right)$ is always less than $\overline{PR}^2$ or $\overline{Pr}^2 \left(\frac{ccxx}{tt}\right)$. And consequently all the Points of the opposite Hyperbola's fall in the Angles formed by their Asymptotes; so that none of them can fall in the adjoining Angles.

COROLLARY II.

92. IF from any two Points M, N, in one Hyperbola, or in the opposite Hyperbola's, there be drawn two right Lines Rr, Kk, perpendicular to the first Axis, and terminating in the Asymptotes: then it is manifest, that the Rectangles $RM \times Mr$, $KN \times Nk$, will be always equal to one another; because each of them is equal to the Square of BC, the half of the second Axis Bb. Whence it appears, that $RM : KN :: Nk : Mr$.

PRO-

PROPOSITION IV.

Theorem.

93. *If the right Lines* Hh, Ll *be drawn from any two Points* (M, N) *of an Hyperbola, or the opposite Hyperbola's, parallel to one another, and terminating in the Asymptotes: I say, the Rectangles* $HM \times Mh$, $LN \times Nl$, *will be equal between themselves.*

We are to prove, that $HM \times Mh = LN \times Nl$.

The right Lines Rr, Kk, being drawn perpendicular to the first Axis Aa; it is then manifest, that the Triangles MRH, NKL, and Mrh, Nkl will be similar; because they are formed by Parallels: And therefore we have $RM : KN :: HM : LN$. And $Nk : Mr ::$ $Nl : Mh$. But * $RM : KN :: Nk : Mr$. Therefore $HM : LN ::$ $Nl : Mh$. And consequently $HM \times Mh = LN \times Nl$. *W. W. D.* * *Art.* 92.

COROLLARY I.

94. If the Line NL, parallel to MH, be suppos'd to pass through the Centre C; or, which is all one, be suppos'd to become the Line CE: Then it is evident, that the two Points L, l, will coincide in the Centre C; and so the Rectangle $LN \times Nl$, will become the Square $\overline{EC}^2$. From whence it appears, that if the right Line CE be drawn from any Point E of one of the opposite Hyperbola's to the Centre C, and then likewise another Line MHb be drawn thro' any Point M of either of those Hyperbola's, parallel to CE, and meeting the Asymptotes in H and b; the Square of CE will be equal to the Rectangle under HM and Mb.

COROLLARY II.

95. If thro' any Point (N) of one of the opposite Hyperbola's be drawn a right Line Ll, terminating in the Asymptotes, and meeting either of those Hyperbola's in some other Point n; then the Parts LN, ln, of that right Line taken between the Points of the Hyperbola, and the Point of Concurrence of the Asymptotes will be equal to one another. For if LN, be called a; Nn, b; nl, c; we shall have $LN \times Nl$ $(ab \mp ac) = HM \times Mb = Ln \times nl$ $(bc \mp ac)$ and so we get $LN\,(a) = ln\,(c)$.

COROLLARY III.

96. If it be suppos'd in the last Corollary, that the Line Nn, terminating in the opposite Hyperbola's, passes thro' the Centre C; that is, if it be suppos'd to become the first Diameter ED; then it is manifest, that the two Points L, l, will coincide in the Centre C; and so NL, will become EC, and nl, CD. From whence it appears, that every first Diameter DE, is divided into two equal Parts by the Centre C.

COROLLARY IV.

97. If two right Lines Mm, Nn, being parallel to one another, be terminated by one Hyperbola, or the opposite Hyperbola's, and meet an Asymptote in the Points H, L; I say, the Rectangles $MH \times Hm$, $NL \times Ln$, will be equal to one another: for if those two Lines be produced (if necessary) until they meet one of the Asymptotes in the Points h, l; then the Parts MH, mh, and NL, nl, will
* Art. 95. be * equal to one another: And therefore, since $HM \times Mh = LN \times Nl$, it follows, that $MH \times Hm = NL \times Ln$.

PROPOSITION V.

Theorem.

FIG. 41. 98. *If thro' any two Points* M, N, *of an Hyperbola, or the opposite Hyperbola's, be drawn two right Lines* MH, NL, *parallel to each other, and terminating in one Asymptote; as likewise two other right Lines* Mh, Nl, *parallel to one another, and terminating in the other Asymptote; I say, the Rectangles* HM × Mh, NL × Nl, *are equal to one another.*

This Proposition is proved in the same manner as the last, the Demonstration being the very same.

COROLLARY I.

FIG. 42. 99. If the right Lines MH, Mh, and NL, Nl, be parallel to the two Asymptotes; then it is manifest, that the Parallelograms $MHCh$, $NLCl$, as likewise the Triangles CHM, CLN, being the Halves of them, are equal to one another; because the Sides of the said Parallelograms about the equal Angles HMh, LNl, are reciprocally proportional.

COROL-

COROLLARY II.

100. THE same Things being premised as in the foregoing Corollary, it is manifest, that $CH \times HM = CL \times LN$; because in this Supposition $Mh = CH$, and $Nl = CL$: that is, if two right Lines MH, NL, be drawn thro' any two Points M, N; in one, or the opposite Hyperbola's, parallel to one of the Asymptotes, and terminating in the other; then the Rectangles $CH \times HM$, $CL \times LN$, will be always equal to one another; and so $CH : CL :: LN : MH$.

COROLLARY III.

101. BEcause the End (A) of the first Axis, is one Point of the Hyperbola; and the Line AB, which cuts one of the Asymptotes CG in G, is parallel to the other Asymptote Cg; therefore * the Rectangle $CH \times HM$ will always be equal to the same Rectangle $CG \times GA$, or to the Square* $\overline{CG}^2$, that is, (according to *Def.* 12.) equal to the Power of the Hyperbola. Then if you call the given Quantity CG, m; and the indeterminate ones, CH, x; HM, y; we shall have always $CH \times HM\ (xy) = \overline{CG}^2\ (mm)$. But because this Property equally extends to all Points of the opposite Hyperbola's, and determines their Position with regard to the Asymptotes; it is evident, that this Equation $xy = mm$ entirely expresses the Nature of the Hyperbola with regard to the Asymptotes.

* *Art.* 100.
* *Art.* 88.

COROLLARY IV.

102. BEcause $HM\ (y)$ is $= \frac{mm}{x}$, it follows, that the more $CH\ (x)$ increases, the more doth $HM\ (y)$ diminish; so that when $CH\ (x)$ becomes infinitely great, $HM\ (y)$ will then be infinitely small; that is, equal to nothing. From whence it appears, that the Hyperbola AM, and its Asymptote CH (being both produced) will accede nearer and nearer to one another; so that at last their Distance will become less than any given Quantity; and yet they will never meet, unless it be at an infinite Distance, to which they can never be produced. The same is to be understood of the other Asymptote Cg.

COROLLARY V.

103. AMong all the Lines that pass through the Centre C, (1.) Those, (as Aa) that fall in those Angles of the Asymptotes next to the Hyperbola's, meet each of the opposite Hyperbola's in only

one Point A, or a; and being produced, will ever after be within the Hyperbola's: for becauſe of the Angles GCA, gCA, and thoſe vertical to them, it is manifeſt, that the Line Aa, recedes more and more from both the Aſymptotes; whereas the oppoſite Hyperbola's
* *Art.* 102. approach nearer * and nearer to them. (2.) Thoſe Lines (as Bb) which fall in the adjoining Angles, alſo formed by the ſaid Aſymptotes, will never meet the oppoſite Hyperbola's altho' infinitely pro-
* *Art.* 91. duced; becauſe none of the Points of the Hyperbola's can fall * in theſe latter Angles.

* *Def.* 9. Whence it appears, * that all firſt Diameters fall in the Angle, form'd by the Aſymptotes, next to the Curve, and the ſecond Diameters in the Angles adjoining to them.

COROLLARY VI.

FIG. 43. 104. IF the Line HM be drawn through any Point H, in one of the Aſymptotes CE, parallel to the other Aſymptote Ce, then that Line HM will meet the Hyperbola in the Point M only; and being continued, will be ever after within the ſame: for the Diſtance from HM to Ce, remains every where the ſame; but the Hyperbo-
* *Art.* 102. la continually comes * nearer and nearer to Ce.

COROLLARY. VII.

105. HENCE if two indefinite right Lines MH, Mb, be drawn thro' any Point M of an Hyperbola, parallel to the Aſymptotes Ce, CE,

1. All the Points of the oppoſite Hyperbola will fall in the Angle
* *Art.* 91. HMb; becauſe they all fall * in the Angle formed by the Aſymptotes, which is included in the Angle HMb.

2. The two Parts of the Hyperbola MN will fall in the Angles, on each Side HMb: ſo that no Point thereof will fall in the Angle vertical to HMb.

3. All Lines, as MF, which fall in the Angle HMb, and being continu'd towards F, do meet the oppoſite Hyperbola in one Point N, and fall within the Curve: becauſe they recede more and more from the right Lines MH, Mb, and conſequently from the Aſymptotes which are parallel to them: But being produced on the other Side of the Point M, they will fall within the Hyperbola paſſing thro' that Point M, and will never after meet the ſame.

4. All Lines, as Ee, which fall in the Angles adjoining to HMb, do meet the two Aſymptotes of the Hyperbola paſſing thro' the Point M; ſo when thoſe Lines fall within one Part of the Hyperbola, they muſt needs meet the ſame in ſome Point (N), ſince they go on to meet the Aſymptote falling without that ~~Part.~~ Point.

Co-

COROLLARY VIII.

106. 1. IF a right Line Ff, be drawn through any Point M of an Hyperbola, meeting one of its Asymptotes in the Point F, and one of the Asymptotes of the opposite Hyperbola in the Point f; and if the said Line be prolonged to N, so that fN be equal to FM: I say, the Point N will be in the opposite Hyperbola. For the Line Ff, falls in the Angle HMb, and consequently meets the opposite Hyperbola in some Point N, as we have demonstrated in the last Corollary. Whence *, &c. * *Art.* 95.

2. If from any Point M of an Hyperbola, there be drawn a right Line Ee, terminating in the Asymptotes, and if you take in it the Part eN equal to EM: I say, the Point N will yet be in that Hyperbola. For drawing MH parallel to one Asymptote Ce, and terminated by the other in H; then if CL be taken in that Asymptote, equal to HE, and the Line LN be drawn parallel to HM; we have demonstrated in *Art.* 104. that the Line LN will meet the Hyperbola in one Point N; and in *Art.* 100. that that Point will be such that CL or $HE : HM :: CH$ or $EL : LN$. From whence it appears, that the Line LN, meets the Hyperbola in the same Point as it meets the right Line Ee. But because HM, LN, are parallel, it is manifest that $eN = EM$, since $CL = HE$. Therefore, &c.

PROPOSITION VI.

Problem.

107. *FROM a given Point* M, *in an Hyperbola, whose Asymptotes* CE, Ce, *are given, to draw the Tangent* DMd; *and demonstrate, that there can be drawn but one only to that Point.* FIG. 43.

Draw the right Line MH from the given Point M parallel to one of the Asymptotes Ce, and terminating in the other (CE) in the Point H; assume the Line HD in CE equal to HC, and draw the right Line DM through the given Point M, meeting the Asymptote Ce in the Point d. I say in the first place, that the Line DMd, will touch the Hyperbola in the Point M.

For because the Triangles CDd, HMD are similar, the Line Dd, terminated by the Asymptotes, is divided by the Point M into two equal Parts, like as CD is in H. And if it were possible for DMd to meet the Hyperbola in some other Point O, then it is manifest, that Od would be * equal to MD, and consequently to Md, that is, the Part equal to the whole; which is impossible: therefore the Line * *Art.* 95.

DMd

D M d cannot meet the Hyperbola in any other Point but *M*. Moreover, if the ſaid Line ſhould fall within the Hyperbola, as the Line *Ee*, it is evident, that it would meet the Curve in ſome other Point
* *Art.* 91. *N*; becauſe it would meet * the Aſymptote falling without the Curve, in the Point *e*. Therefore it is plain, that the Line *D d*, meets the Hyperbola only in the Point *M*, and doth not fall within the ſame, that is, the ſaid Line touches the Hyperbola in the Point *M*.

2. I ſay, there is no Line but *D M d* only can touch the Hyperbola in the Point *M*: For if *H E* be taken in one of the Aſymptotes *CE*, either greater or leſs than *H D*, and if the right Line *E M* be drawn from the Point *M*, meeting the other Aſymptote *C e* in the Point *e*: Then becauſe *M H*, *C e* are parallel, it is manifeſt, that *M E* will be greater or leſs than *M e*; ſince *H E* was aſſum'd greater or leſs than *H D* or *H C*. Now this being premis'd, if the Point *N* be taken in the greater Part *M e*, ſo that *N e* be equal to *M E*; then it is evident,
* *Art.* 106. that the ſaid Point *N* will be * in the Curve, and ſo the Line *E e* will not touch the ſame in the Point *M*. Which was the ſecond Thing to be demonſtrated.

SCHOLIUM.

108. It has been demonſtrated in *Art.* 102. that the more *CH* increaſes, the more doth *H M* diminiſh; ſo that when *CH* becomes infinitely great, *H M* will become infinitely ſmall, or nothing. But when *C H* is infinitely great, then *H D* (being equal thereto) will be ſo likewiſe; and conſequently the Lines *M D*, *H D*, meeting one another at an infinite Diſtance, being taken as Parallels, will fall in each other; becauſe the Points *M* and *H* will then coincide: that is, the Aſymptote *CE* being infinitely produced, (as alſo the Hyperbola) may be taken for a Tangent to the Hyperbola, in the Extremity thereof. The ſame may be ſaid of the other Aſymptote *C e*, which may be eſteem'd as touching the ſame Hyperbola in the other Extremity thereof.

Hence it appears, that the two Aſymptotes may be taken as infinite Tangents touching the oppoſite Hyperbola's in the Extremities thereof.

COROLLARY I.

109. SINCE there is but one Line *D M d*, only which terminating in the Aſymptotes, is divided into two equal Parts in the Point *M*; it follows, if any right Line *D M d*, terminating in the Aſymptotes of an Hyperbola, meets the ſame in the Point *M*, dividing

ding that right Line into two equal Parts; then that Line DMd, will touch the Hyperbola in the Point M. And contrariwiſe, if a right Line DMd, terminating in the Aſymptotes of an Hyperbola, touches the ſame in M; then will that Line be biſected in the Point M.

COROLLARY. II.

110. IF any firſt Diameter MCm be drawn thro' M the Point of Contact of any Tangent DMd, terminating in the Aſymptotes CL, Cl, of an Hyperbola; and if Ee be drawn thro' the Point m, wherein MCm meets the oppoſite Hyperbola, parallel to the Tangent Dd, and terminating in the Aſymptotes in the Points E, e; then will the Line Ee be a Tangent to the Hyperbola in the Point m. For the Triangles CMD, CmE, will be ſimilar and equal, becauſe * CM is equal to Cm: therefore the Line mE will be equal to MD. We prove after the ſame manner (becauſe the Triangles CMd, Cme are ſimilar and equal) that me is equal to Md: therefore the Line Ee is divided in the Point m into two equal Parts; becauſe Dd is ſo divided in the Point M; and conſequently the Line Ee, touches * the Curve in the Point m. FIG. 44. * Art. 96. *Art. 109.

Hence it appears, that the Tangents Dd, Ee, paſſing thro' the Extremities of any firſt Diameter Mm, are parallel to one another; and alſo equal, when they are terminated by the Aſymptotes.

DEFINITIONS.

13.

If there are two Diameters Mm, Ss, whereof one, as Ss, is parallel to the Tangents paſſing through the Extremities of the other Mm; and terminated beſides in S, s, by the right Lines MS, Ms, drawn through the Point M, one End of the Diameter Mm, parallel to the Aſymptotes; the ſaid two Diameters Mm, Ss are call'd together *Conjugate Diameters*. FIG. 44.

14.

Right Lines drawn from Points of the oppoſite Hyperbola's parallel to one of the Conjugate Diameters, and terminating in the other, are called *Ordinates* to that Diameter. So NO is an Ordinate to the Diameter Mm.

15.

If a third Proportional be taken to the two Conjugate Diameters, then the ſame will be the *Parameter* of that Diameter, which is the firſt Term of the Proportion.

Co-

Corollary. I.

111. THE thirteenth *Definition* hath Relation to the two Axes; because, according to *Art.* 84. the second Axis is parallel to the Tangents passing thro' the Ends of the first; and moreover (by *Def.* 11.) is terminated by two right Lines drawn from one End of the first Axis parallel to the Asymptotes: Whence it appears, that the two Axes may be taken as two Conjugate Diameters, being at right Angles with one another.

Corollary II.

112. BEcause the Diameter SCs, is parallel to the Tangent DMd, passing through M, one End of the Diameter MCm; and since that Tangent meets the two Asymptotes (CD, Cd) of the Hyperbola, passing through the Point M: therefore the Diameter SCs falls in the Angles adjacent to the Angle DCd, form'd by the Asymptotes of the Hyperbola; and so it will be a second Diameter.

Hence it appears, that among any two Conjugate Diameters, as MCm, SCs, there is always a first Diameter Mm, and a second Diameter Ss.

Corollary. III.

113. THE second Diameter SCs, is divided into two equal Parts by the Centre C, and is also equal to the Tangent DMd, which passing thro' M, one End of the first Diameter Mm, being a Conjugate to SCs, does terminate in the Asymptotes. For since MS, Cd, and Ms, CD, are parallel; it is manifest, that CS is equal to
*Art. 109. Md, and Cs to MD. But DMd, is divided * into two equal Parts in M the Point of Contact. Therefore, &c.

Corollary IV.

114. IF two Conjugate Diameters Mm, Ss be given, and it is known which of them is the first Diameter; then you may have the Asymptotes CD, Cd, in drawing right Lines parallel to the two right Lines MS, Ms, (drawn from M the Extremity of the first Diameter Mm, to S, s, the two Ends of the second.)

And contrariwise, if the two Asymptotes CD, Cd, of an Hyperbola be given, together with some Point M thereof; you may find the two Conjugate Diameters MCm, SCs, by drawing the Line MH parallel to one of the Asymptotes (Cd), meeting the other Asymptote

CD in H, and producing the same to S, so that HS be equal to HM; and then drawing the right Lines CM, CS; for if MD be drawn parallel to CS, it is evident, (since the Triangles CHS, MHD, are similar,) that HD is equal to HC, because MH is equal to HS; and so MD touches * the Curve in M: Therefore, by *Def.* 13. the Lines CM, CS, are two Semi-Conjugate Diameters. *Art. 107.

Hence, if two Conjugate Diameters Mm, Ss be given in Position and Magnitude, and if it be known which of them is the first Diameter, then the two Asymptotes CD, Cd, together with the Point M, one Point of the opposite Hyperbola's, is had.

And contrariwise, the Asymptotes CD, Cd of an Hyperbola being given, together with one Point M of the same; we have the two Conjugate Diameters Mm, Ss thereof given both in Position and Magnitude; as likewise we know which of them is the first Diameter, being that passing thro' the given Point M.

COROLLARY IV.

115. ANY second Diameter SCs, being given in Position, if the Magnitude thereof be requir'd, as also the first Diameter Mm, the Conjugate to SCs, you must draw the right Line Ll, any where in the Angle form'd by the Asymptotes, parallel to the second Diameter, and terminated by the Asymptotes in the Points L, l; and then through O the middle of Ll, you must draw the first Diameter CO, meeting the Hyperbola in one Point M. This being done, if the right Lines MS, Ms, be drawn from the Point M parallel to the Asymptotes: It is manifest, by *Def.* 13. that the Points S, s, wherein those Parallels meet the second Diameter SCs given in Position, do determine the Magnitude thereof; as also that the first Diameter MCm is the Conjugate thereto. For if the Line Dd be drawn thro' the Point M, parallel to Ll, and terminating in the Asymptotes; then the said Line will be bisected in the Point M, because Ll is so in the Point O; and therefore Ll will touch * the Curve in the Point M.

Hence it is evident, that any second Diameter SCs being given in Position, the Magnitude thereof is so determin'd, as that it cannot vary; as likewise the Magnitude and Position of the first Diameter Mm, which is a Conjugate thereto.

COROLLARY V.

116. ANY second Diameter SCs, being given in Position and Magnitude, together with the Parameter, and the Position of the Ordinates to it; then it will not be difficult to find the Position

and Magnitude of the first Diameter MCm, being the Conjugate to SCs, as also its Parameter. For through the Centre C draw an indefinite right Line parallel to the Ordinates to the Diameter Ss, and denote two Points M, m in this Line equally distant each way from the Centre C; so that Mm be a mean Proportional between the second Diameter Ss, and the Parameter thereof. Then if a third Proportional to the two Lines Mm, Ss be found, it is manifest, by *Def.* 14, and 15. that Mm will be the first Diameter, being the Conjugate to Ss, and the Parameter thereof will be that third Proportional.

PROPOSITION VII.

Theorem.

FIG. 44. 117. *THE Square of any Ordinate* (ON) *to the first Diameter* Mm, *is to the Rectangle under* MO, Om, *the Parts of that Diameter produced; as the Square of its Conjugate Diameter* Ss, *to the Square of that first Diameter* Mm.

We are to prove, that $\overline{ON}^2 : MO \times Om :: \overline{Ss}^2 : \overline{Mm}^2$.

If the right Line Dd, be drawn through one End (M) of the first Diameter Mm, parallel to the second Diameter Ss, and terminating in the Asymptotes; then, (by *Def.* 13.) that Parallel will touch the

* *Art.* 109. Curve in the Point M, and so will be * bisected by that Point: therefore if the Ordinate ON (which by *Def.* 13. is parallel to the Diameter Ss) be produced both ways from the Diameter Mm, the same will meet the Asymptotes in two Points L, l, each way equally distant from the Point O. This being premis'd, call the given Quan-

* *Art.* 113. tities CM, or Cm, t; CS, or Cs, or * MD, or Md, c; and the indeterminate Quantities CO, x; ON, y; then because the Triangles CMD, COL, are similar, we have this Proportion, $CM\,(t) : MD\,(c) :: CO\,(x) : OL$ or $Ol = \frac{cx}{t}$. Whence LN or $LO \pm ON = \frac{cx}{t} \pm y$, and Nl or $Ol \mp NO = \frac{cx}{t} \mp y$; and therefore $LN \times Nl = \frac{ccxx}{tt} - yy$

* *Art.* 90. $=$ * $DM \times Md = cc$. Whence it follows, that $\overline{ON}^2\,(yy) : MO \times Om\,(xx - tt) :: \overline{Ss}^2\,(4cc) : \overline{Mm}^2\,(4tt)$. Because by multiplying the Means and Extremes, we have $4ttyy = 4ccxx - 4cctt$; that is (by dividing by $4tt$, and transposing) the same Equation $\frac{ccxx}{tt} - yy = cc$, as at first. *W. W. D.*

Ag-

A General COROLLARY.

118. HENCE it is manifest, that what has been demonstrated * in *Prop.* 2. with regard to the two Axes Aa, Bb, extends it self, by means of this Proposition, to any two Conjugate Diameters Mm, Ss. And because the 80th, 81st, 82d, 83d, 84th and 85th Articles arise from the second Proposition, and are of equal Force, whether the Angle ACB be a right one or not; therefore it follows, that if the Lines Aa, Bb, instead of the two Axes, be supposed in these Articles to be any two Conjugate Diameters; the said Articles will yet be true according to this Supposition, for their Demonstration remains always the same; and there is nothing more requir'd to make this appear, but reading them over again, and using the Word Diameter for Axis. * *Art.* 79.

PROPOSITION VIII.

Theorem.

119. *IF* DE, FG *be any two Tangents to the Hyperbola* MA, *terminating in the Asymptotes, and cutting one another in the Point* O; *I say, the Sides of the Triangles* CDE, CFG, *about the common Angle* C, *are reciprocal proportional.* FIG. 45.

We are to prove, that CD : CF :: CG : CE.

Draw the Lines MH, AL through the Points of Contact M, A, parallel to the Asymptote CG; then it is manifest, (because the Triangles CDE, HDM are similar) that CD is the Double of CH, and CE the Double of HM; since DE is * the Double of DM. And because the Triangles CFG, LFA are similar, CF is the Double of CL, and CG the Double of LA, because FG is the Double of FA. But * $CH : CL :: LA : HM$. And therefore, if the Double of each Term be taken, we shall have $2CH$ or $CD : 2CL$ or $CF :: 2LA$ or $CG : 2HM$ or CE. *W. W. D.* * *Art.* 109. * *Art.* 106.

COROLLARY.

120. IT follows from this Proposition, that the right Lines DG, FE, are parallel to one another: Whence it is manifest,

1. The Triangles CDE, CFG, are equal: For the Triangles FDE, FGE, having the same Base FE, and being between the same Parallels DG, FE, are equal; and therefore, if the same Triangle CFE be added to both the Triangles CDE, CFG, there will be form'd the Triangles CDE, CFG, which shall be equal to one another.

2. The Line DE is divided in the same Proportion in the Points M, O, as the Line FG is in the Points A and O. For if the right Line MA be drawn thro' the Points of Contact, then it is manifest, that this Line will be parallel to the two right Lines DG, FE; because it bisects the right Lines DE, FG, included between those Parallels.

PROPOSITION IX.

Theorem.

FIG. 46, and 47. 121. *If through any Point* M *in an Hyperbola, be drawn an Ordinate* (MP) *to any one of its Diameters* Aa, *and if the Tangent* MT *be also drawn meeting that Ordinate in* T; *I say,* $CP : CA :: CA : CT$. *Observing that the Points* P, T *fall on the same Side the Centre* C, *when the Line* Aa *is a first Diameter; and on both Sides, when it is a second Diameter.*

FIG. 46. *Case* 1. When the Line Aa is a first Diameter, produce the Tangent MT, meeting the Asymptotes CD, CG, in the Points D, E; and produce the Ordinate PM, meeting the Asymptote CD in the Point N; also draw the Line AK through the Point A parallel to DE, meeting the Asymptote CG in the Point K; and likewise draw the Tangent

* Def. 14. FG, terminating in the Asymptotes (which will be * parallel to PM) and meeting the other Tangent DE in the Point O.

This being laid down, AP is to AC, or FN to FC, in a Ratio

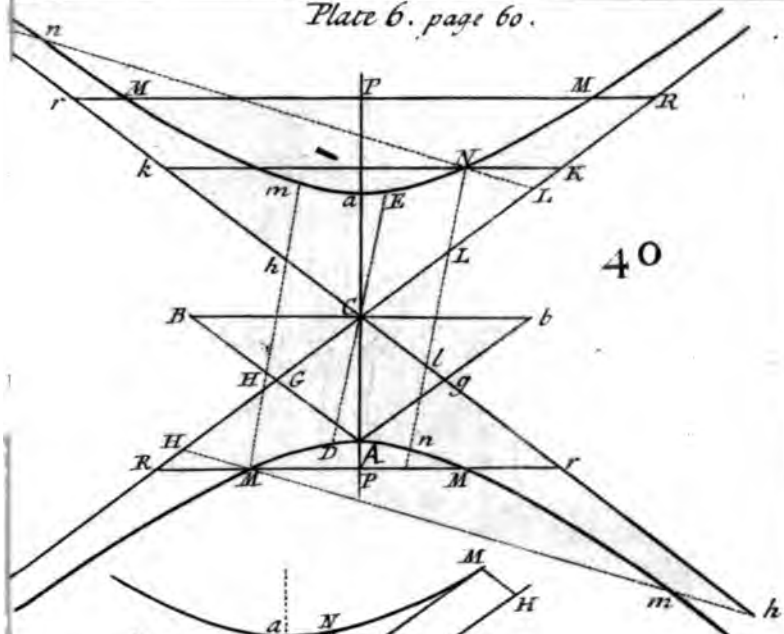
*Art. 120. compounded of FN to FD, or of OM to OD, or * of OA to OG,

*Art. 120. or of EK to EG, and of FD to FC, or * of EG to EC. But MT is to TC, or KE to EC, in the Ratio compounded of EK to EG, and of EG to EC: therefore $AP : AC :: AT : TC$. Because the Ratio's compounded of those two Ratio's are the same; and consequently $AP + AC$, or $CP : CA :: AT + TC$, or $CA : CT$. *Which was in the first place to be demonstrated.*

FIG. 47. *Case* 2. When the Line Aa is a second Diameter, draw the Line CK through the Centre C parallel to the Ordinate PM, meeting the Hyperbola in the Point B, and the Tangent MT, in the Point R, and draw the Line MK through the Point of Contact M parallel to Aa; then it is manifest, that CB will be a first Semi-Diameter, the Conjugate to the Second Aa; and so MK will be an Ordinate to that Diameter.

This being premised, if the given Quantities CA, or Ca be called t; CB, c; and the indeterminate Quantities CP or MK, x; PM or CK, y; then from what has been demonstrated in *Case* 1. we have $CR = \frac{cc}{y}$; and therefore RK or $CK - CR = \frac{yy-cc}{y}$. But the similar Triangles KRM, CRT, give this Proportion $KR \left(\frac{yy-cc}{y}\right) : RC \left(\frac{cc}{y}\right)$

$:: MK$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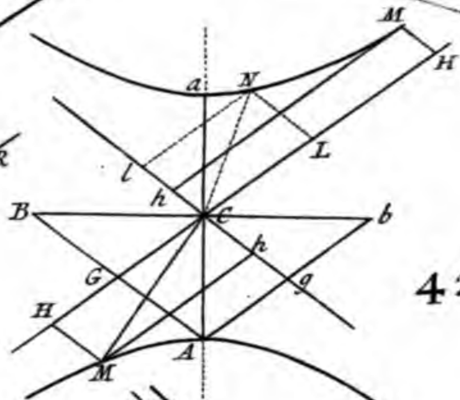
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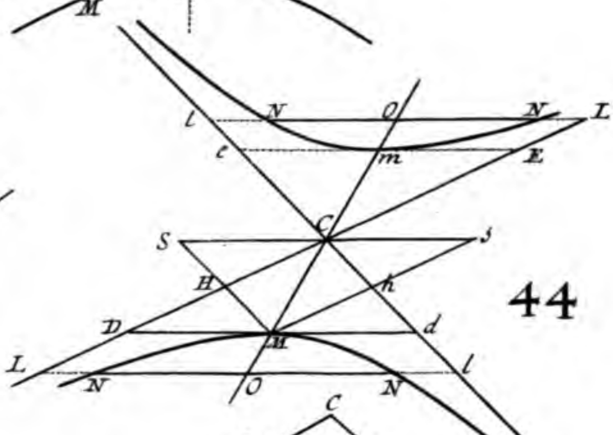
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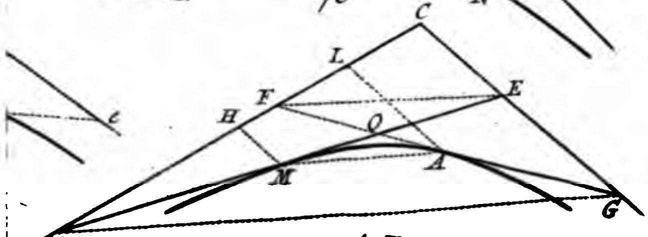
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$:: MK(x) : CT = \frac{cxx}{yy-cc} = \frac{tt}{x}$, (by substituting $\frac{ccxx}{tt}$ for its Value $yy-cc$ (because $yy = {}^{*}\frac{ccxx}{tt} + cc.$) That is, $CP : CA :: CA : CT$. *W.W.D.* * *Art.* 80, *and* 118.

PROPOSITION X.

Theorem.

122. *IF through any Point* M *in an Hyperbola, whose Centre is* C, *there be drawn an Ordinate* MP *to one of the Axes* Aa, *as also the Perpendicular* MG *to the Tangent* MT *passing through* M: *I say,* CP *will always be to* PG, *in the given Ratio of the Axis* Aa *to the Parameter thereof.* FIG. 48, and 49.

For call the Semi-Axis CA or Ca, t; and the indeterminate Quantities CP, x; PM, y; then we shall have * $CT = \frac{tt}{x}$; and therefore * *Art.* 121.

$PT = \frac{xx \mp tt}{x}$, according as Aa, is the first or second Axis. But the right-angled similar Triangles TPM, MPG, give this Proportion $TP\left(\frac{xx \mp tt}{x}\right) : PM(y) :: PM(y) : PG = \frac{xyy}{xx \mp tt}$. Whence we get this Proportion, *viz.* $CP(x) : PG\left(\frac{xyy}{xx \mp tt}\right) :: \overline{CP}^2 \mp \overline{CA}^2 (xx \mp tt) : \overline{PM}^2 (yy)$. Since by multiplying the Means and Extremes, the same Product xyy arises. But $\overline{CP}^2 \mp \overline{CA}^2$ is to $\overline{PM}^2$, as * the Axis Aa to the Parameter thereof. Therefore CP is to PG likewise in the same Ratio. *W.W.D.* * *Art.* 82.

PROPOSITION XI.

Theorem.

123. *IF the right Lines* MF, Mf, *be drawn from any Point* M, *in an Hyperbola, to the two Foci* F, f: *I say, the Tangent* MT, *passing through that Point* M, *does divide the Angle* FMf *into two equal Parts.* FIG. 50.

For draw the Lines FD, fd, perpendicular to the Tangent MT; likewise draw the first Axis Aa, passing through the two Foci F, f, and meeting the Tangent in T; as likewise the Ordinate MP to that Axis: Then call the given Quantities CA or Ca, t; CF or Cf, m; and the indeterminate Quantity CP, x. This being done, we have MF * * *Art.* 78. $\left(\frac{mx}{t} - t\right) : Mf\left(\frac{mx}{t} + t\right) :: TF$ or $CF(m) - CT^{*}\left(\frac{tt}{x}\right) : Tf$ or Cf * *Art.* 121.

(m)

$(m) + CT\left(\frac{tt}{x}\right)$. since by multiplying the Means and Extremes, the same Product arises. But the right-angled similar Triangles TFD, Tfd, give this Proportion, $TF : Tf :: FD : fd$; therefore the Hypothenuse (MF) of the right-angled Triangle MDF, will be to the Hypothenuse Mf, of the right-angled Triangle Mdf, as the Side DF is to the Side df; and consequently these two Triangles will be similar. Therefore the Angles FMD, fMd, which are opposite to the Homologous Sides DF, df, will be equal to one another. *W. W. D.*

COROLLARY.

124. HENCE it is manifest, that the Tangent MT being infinitely produced both ways from (M) the Point of Contact, leaves the Hyperbola AM entirely next to its Focus F. And because this every where happens, let the Point M be taken where it will in the Curve: Therefore it is manifest, that the Hyperbola being extended never so much, is Concave next to its Focus F.

PROPOSITION XII.

Theorem.

FIG. 51. 125. *THE Difference of the Squares of any two Conjugate Diameters* M m, S s, *is equal to the Difference of the Squares of the two Axes* A a, B b.

We are to prove, that $\overline{CS}^2 - \overline{CM}^2 = \overline{CB}^2 - \overline{CA}^2$, *or* $\overline{CM}^2 - \overline{CS}^2 = \overline{CA}^2 - \overline{CB}^2$.

If the right Lines MS, AB are drawn, they will be * parallel to one of the Asymptotes Cg; as also cut into equal Parts by the other Asymptote CG in the Points H, G; because * the Lines Ms, Ab, are parallel to the Asymptote CG, and the second Diameters Ss, Bb, are * divided by the Centre C into two equal Parts; therefore, if the right Lines AF, BE, ML, SK, be drawn perpendicular to the Asymptote CG, the Triangles GAF, GBE, and HML, HSK, will be formed, which shall be similar and equal. This being premis'd, call the given Quantities CG or * GA, m; GE or GF, a; AF or BE, b; and the indeterminate Quantities CH, x; HM, y; then we have $CE = m + a$, $CF = m - a$; $\overline{CE}^2 + \overline{EB}^2$, or $\overline{CB}^2 = mm + 2am + aa + bb$, $\overline{CF}^2 + \overline{FA}^2$, or $\overline{CA}^2 = mm - 2am + aa + bb$: And therefore $\overline{CB}^2 - \overline{CA}^2 = 4am$. But the similar Triangles GAF, HML, give

* Def. 11, and 13.
* Def. 11, and 13.
* Art. 113.
* Art. 88.
* Art. 121.

(n)

give this Proportion, $GA\,(m) : AF\,(b) :: HM\,(y) : ML$ or $KS = \frac{by}{m}$. And $GA\,(m) : GF\,(a) :: HM\,(y) : HL$ or $HK = \frac{ay}{m}$. Therefore $CK = x + \frac{ay}{m}$, $CL = x - \frac{ay}{m}$; $\overline{CK}^2 + \overline{KS}^2$ or $\overline{CS}^2 = xx + \frac{2axy}{m} + \frac{aayy}{mm} + \frac{bbyy}{mm}$, $\overline{CL}^2 + \overline{LM}^2$ or $\overline{CM}^2 = xx - \frac{2axy}{m} + \frac{aayy}{mm} + \frac{bbyy}{mm}$. And therefore, $\overline{CS}^2 - \overline{CM}^2 = \frac{4axy}{m} = 4am$, by putting * mm for xy. And consequently $\overline{CS}^2 - \overline{CM}^2 = \overline{CB}^2 - \overline{CA}^2$. *W. W. D.* * *Art.* 101.

If the Angle GCg, formed by the Aſymptotes, ſhould be acute; whereas in this Figure, and the Reaſoning appropriated thereto, it is obtuſe; then CF would be greater than CE; and it would be proved after the ſame manner, that $\overline{CM}^2 - \overline{CS}^2 = \overline{CA}^2 - \overline{CB}^2$. But if the Angle GCg, form'd by the Aſymptotes, was a right Angle; then it is manifeſt, that the Lines AB, MS, would be perpendicular to the Aſymptote CG; and ſo the two Semi-Conjugate Diameters CM, CS, would be equal to one another, like as the two Semi-Axes CA, CB. But becauſe the Difference of the two Conjugate Diameters Mm, Ss, is nothing; as likewiſe the Difference of the two Axes Aa, Bb; therefore it follows, that this Propoſition is true in all its Caſes.

COROLLARY.

126. HENCE it is manifeſt, that any firſt Diameter Mm, is leſs, greater than, or equal to the ſecond Diameter Ss, being the Conjugate thereto; according as the Angle GCg, formed by the Aſymptotes, is obtuſe, acute or right.

DEFINITION.

16.

Two oppoſite Hyperbolas are called *Equilateral*, when any two of their Conjugate Diameters are equal to one another; or elſe when the Angle form'd by their Aſymptotes is a right Angle.

COROLLARY.

127. IF from any Point M in an equilateral Hyperbola, there be drawn any Ordinate (MP) to either of the Diameters, as Aa, then we ſhall have * $\overline{MP}^2 = \overline{CP}^2 \mp \overline{CA}^2$; *viz.* —, when Aa is a firſt Diameter and +, when it is a ſecond Diameter. For the Conjugate Diameter to Aa, will be * always equal to it. FIG. 51. * *Art.* 81, *and* 118. * *Art.* 126.

PRO-

PROPOSITION XIII.

Problem.

FIG. 53, 54, and 55. 128. *ANY two Conjugate Diameters being given, and knowing which of them is the first Diameter; or, which comes * to the same thing,*
* Art. 114. *the Asymptotes* CD, CF *of an Hyperbola being given, together with any Point* (M) *of the Curve: to draw two Conjugate Diameters* Aa, Bb, *that shall make an Angle with each other, equal to an Angle given.*

In any Circle whose Centre is o, draw the Chord df, so that the Angle in the Segment dcf be equal to the Angle DCF form'd by the Asymptotes; and draw the Line ec through e the Middle of the Chord df, making the Angle dec, or fec with that Chord equal to the given Angle; and thro' the Point c, wherein the Line ec meets the Arc dcf, draw the right Lines cd, cf. This being done, assume CD, CF, in the Asymptotes equal to the Chords cd, cf; then if DF be drawn, and the *second Diameter* Bb be drawn parallel to the same; and the first Diameter Aa, through E the middle Point; I say, the two Diameters Aa, Bb, make an Angle with one another equal to the given Angle, and they are Conjugate Diameters.

For by Construction, the Angle dcf, is equal to the Angle DCF form'd by the Asymptotes; and consequently the Triangles DCF, dcf, and DCE, dce, are equal and similar; therefore the Angle BCa made by the Diameters Aa, Bb, will be equal to the Angle DEC or dec, which was made equal to the given Angle. Moreover, if through the Point A, one End of the first Diameter Aa, there be drawn a Parallel to DF, it is manifest, that this Parallel will be divided into two equal Parts by the Point A, because DF is so divided
* Art. 109. in the Point E; and so * that Parallel will touch the Curve in A;
* Def. 13. therefore * Aa, Bb, are Conjugate Diameters.

Now to determine the Magnitudes of the said two Conjugate Diameters, draw the Line MKL thro' the given Point M parallel to the Diameter Aa, meeting one of the Asymptotes (CD) in the Point K; and the other Asymptote CF, (produced beyond the Centre C) in the Point L: This being done, if CA be taken a mean Propor-
* Art. 94. tional between KM, ML; then it is * manifest, that the Point A will be one End of the first Diameter Aa; and so if the Lines AB,
* Def. 13. Ab, are drawn parallel to the Asymptotes CF, CD, those * Parallels will determine the Magnitude of the second Diameter Bb, by their Points of Concurrence B, b.

Because there can be drawn two different Lines ec, ec, making the Angles dec, fec each way with the Chord df, equal to the given Angle, if it be not a right Angle; therefore we can find always two diffe-

different Pair of Conjugate Diameters (*Aa*, *Bb*) which will anſwer the Problem, as they be ſeen in *Fig.* 54, and 55. But it muſt be noted, that the Conjugate Diameters *Aa*, *Bb*, of *Fig.* 55. have the ſame Poſition with reſpect to the Aſymptote *CF*, as thoſe of *Fig.* 54. have to the other Aſymptote *CD*; and their Magnitudes will remain the ſame in theſe two different Poſitions. For,

1. If the Line *oe* be drawn from the Centre *o* to *e*, the Middle of the Chord *df*; this Line will be perpendicular to that Chord: and conſequently the Angles *oec*, *oec*, will be equal; therefore drawing the Radii *oc*, *oc*, the Triangles *oec*, *oec*, which have the Side *oe* common, the Angles *oec*, *oec*, and the Sides *oc*, *oc*, equal to one another, will have alſo their third Sides *ec*, *ec*, equal: therefore the Triangles *fec*, *dec*, which have the Sides *ef*, *ed*, and *ec*, *ec*, as alſo the Angles *fec*, *dec*, equal, will be equal and ſimilar: And ſo it appears, that the Angle *ecf*, or *ECF*, of *Fig.* 55. is equal to the Angle *ecd*, or *ECD*, of *Fig.* 54. and conſequently the Poſition of the Diameter *Aa*, in *Fig.* 55. with regard to the Aſymptote *CF*, is the ſame as the Poſition of the Diameter *Aa*, in *Fig.* 54. with regard to the other Aſymptote *CD*.

2. If the Line *Ml* be drawn, in *Fig.* 55. making the Angle *MlC*, with the Aſymptote *CF*, (produced) equal to the Angle *MLC*, or *ECF* of *Fig.* 54. then it is plain, that the Lines *Ml*, *Mk*, in *Fig.* 55. will be equal to the Lines *ML*, *MK*, of *Fig.* 54. becauſe the Poſition of the Point *M*, in reſpect of the Aſymptotes, is ſuppos'd to be the ſame in both Figures. But the Angle *MlL*, being the Complement of the Angle *MlC*, in *Fig.* 55. or of *ECF*, in *Fig.* 54. is equal to the Angle *MKk*, being the Complement of the Angle *ECD*, in *Fig.* 55. or of *ECF*, in *Fig.* 54. and conſequently (in *Fig.* 55.) the two Triangles *LMl*, *kMK*, having the Angle at *M* common, and the Angles at the Point *l*, *K*, equal, will be ſimilar; and ſo $LM : Ml :: kM : MK$. Therefore $LM \times MK = lM \times Mk$, or $LM \times MK$, in *Fig.* 54. Hence it appears, * that *CA*, *CA* the Halves of the firſt Diameters, in *Fig.* 54 and 55. are equal. The ſame may be ſaid of the Diameter *Bb*; becauſe the Poſition and Magnitude thereof depends on the Poſition and Magnitude of the firſt Diameter *Aa*, to which it is the Conjugate. * *Art.* 94.

Becauſe there can be drawn but one Line *ee*, making an Angle either way with the Chord *df*, equal to the given Angle, when it is a right one; therefore there can be but two Conjugate Diameters *Aa*, *Bb*, making right Angles with one another that will anſwer the Problem, and theſe * will be the Axes. But becauſe the Triangle *dcf*, or *DCF*, is then Iſoſceles, the firſt Axis *Aa* will biſect the Angle *DCF* formed by the Aſymptotes; and ſo there is nothing more requir'd for finding the Poſition of the two Axes, but drawing the two right Lines *Aa*, *Bb*, FIG. 55, and 57. * *Art.* 111.

K perpen-

perpendicular to one another; one of which, as $A a$, bisects the Angle DCF, formed by the Asymptotes; for afterwards their Magnitude may be determined, as is directed for finding the Magnitudes of the Conjugate Diameters.

The two Axes may be found otherwise thus: Draw MH through the Point M, parallel to CF one of the Asymptotes, and terminating in the Point H by the other Asymptote CD. And in the Asymptote CD, assume CG, a mean Proportional between CH, HM; and draw AB thro' the Point G parallel to CF, so that each of its Parts GA, GB, be equal to CG. Then it is manifest, that the Lines CA,
*Art. 101, and 88. CB, will be * the two Semi-Axes both in Position and Magnitude.

COROLLARY.

129. IT is now evident, 1. That there are but two Conjugate Diameters that cut each other at right Angles; and so there can be but two Axes. 2. There can be but two different Pair of Conjugate Diameters making an Angle with each other equal to a given Angle, when this Angle is not a right one; and the two first Diameters of these two Pair have the same Position to one Asymptote, as the two others have to the other Asymptote; and so they are alike situate on both Sides of the two Axes, since the two Axes bisect the Angles form'd by the Asymptotes: And finally, their Magnitudes remain the same in both those different Positions.

PROPOSITION XIV.

Problem.

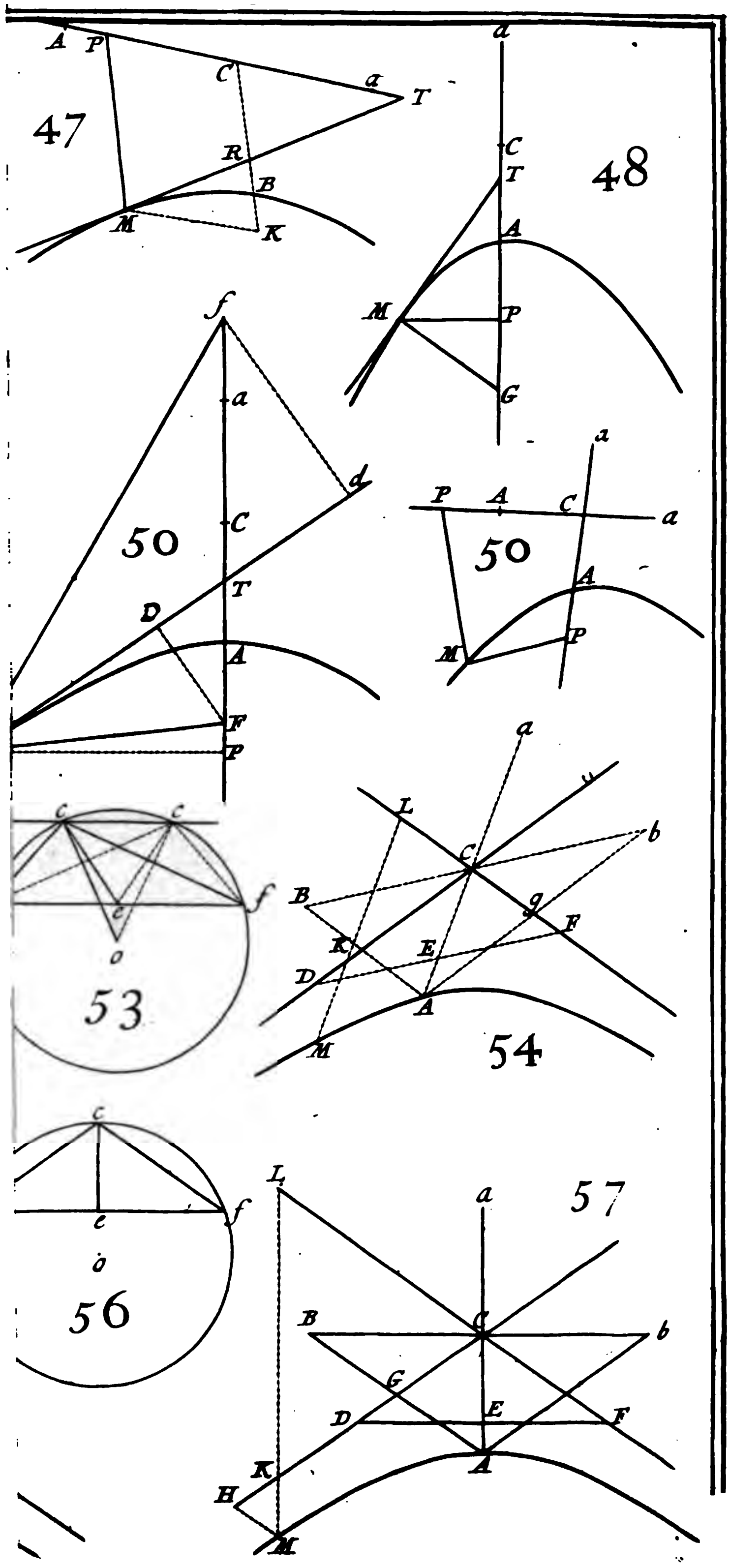
130. *ANY two Conjugate Diameters being given, and which of the two*
*Art. 114. *is the* first Diameter *being known; or, which * is the same thing, the Asymptotes of two opposite Hyperbola's being given, together with any one of their Points, to describe the said Hyperbola's by a continued Motion.*

The First Way.

Find the two Axes, as is directed in the last Proposition, and then describe the opposite Hyperbola's by Article 76.

Second Way.

FIG. 58. Let $A a$, $B b$, be any two given Conjugate Diameters, whereof $A a$ is the first; or else let CG, CG, be two Asymptotes given, together with the Point A, through which one of the opposite Hyperbola's passes.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passes. Draw AG through the given Point A, parallel to one of the Asymptotes Cg, and terminating in the other in G; then move the right Line HK, (equal to CG) along the Asymptote CG, both ways indefinitely produced; so that one End H thereof carries along with it the Line HM parallel to the Asymptote Cg, and the other End K, the right Line KA, moveable about the fixed Point A. Then I say, the Point (M), being the continual Intersection of the right Lines AK, HM, will by this Motion describe the two opposite Hyperbola's sought.

For because the Triangles KHM, KGA, are similar, we have always this Proportion, KH or $CG : HM :: KG$ or $CH : GA$. And therefore $CH \times HM = CG \times GA$. Therefore the Point M will be * in the Hyperbola passing through the given Point A, or in the opposite Hyperbola, and whose Asymptotes are the given right Lines CG, Cg. * *Art.* 101.

PROPOSITION XV.

Problem.

131. *THE same Things being given, as in the last Proposition, to describe the opposite Hyperbola's by finding many Points thereof.*

First Way.

Let CD, CE, be the given Asymptotes, and A the given Point. FIG. 59. Thro' this Point A draw any Number of Lines DE, DE, DE, &c. terminating in the Asymptotes; and in them assume EM, EM, EM, &c. each equal to its Correspondent AD, AD, AD, &c. Then it is * manifest, 1. That the Points M, M, M, &c. will be in the Hyperbola passing thro' the Point A, when the Points E, E, E, &c. fall below the Centre. 2. That the right Lines CD, CE, are the Asymptotes of the Hyperbola's: Therefore, if two Curves be drawn through all the Points M, M, M, &c. falling in the opposite vertical Angles; those two Curves will be the opposite Hyperbola's sought. * *Art.* 106.

Second Way.

Let the Lines Aa, Bb, be two given Conjugate Diameters, whereof FIG. 60. Aa is the 2d *Diameter*. Assume any Number of little equal Parts CE, CE, CE, &c. of any Magnitude at Pleasure, in the Semidiameter CB indefinitely produced towards B; and through that Point E, which is nearest to the Centre C, draw the Line EP parallel to BA. This being done, in the second Diameter Aa, both ways continued, assume

the ſmall Parts CP, PP, &c. each equal to CP, as many in Number as the Parts CE, EE, EE, &c. and draw CD perpendicular and equal to CB; then if the Lines MPM, MPM, MPM, &c. be drawn parallel to the firſt Diameter Bb, and in each of them you aſſume, both ways from the Point P, the Parts PM, PM, each equal to its Correſpondent ED. I ſay, the two Curve Lines paſſing thro' all the Points M, M, M, &c. thus found will be the two oppoſite Hyperbola's ſought.

For call the given Quantities CA, t; CB or CD, c; and the indeterminate Quantities CP, x; PM, y; then the ſimilar Triangles CAB, CPE give this Proportion, $CA\ (t) : CB\ (c) :: CP\ (x) : CE = \frac{cx}{t}$. And ſince the Triangle ECD is right-angled at C (ſuppoſing every Hypothenuſe ED to be drawn, which, for avoiding Confuſion, we have omitted in the Figure) the Square $\overline{ED}^2$ or $\overline{PM}^2\ (yy)$ is $=$

* Art. 81, and 118. $\overline{CE}^2 \left(\frac{ccxx}{tt}\right) + \overline{CD}^2\ (cc)$. Therefore the Line PM will be * an Ordinate to the ſecond Diameter Aa, having the firſt Diameter Bb, the Conjugate thereto. And becauſe the ſame Demonſtration extends to every of the Lines PM, ſince CP is always to its Correſpondent PE, in the Ratio of CA to CB: Therefore, &c.

FIG. 61. When the Conjugate Diameters Aa, Bb, are equal to one another,
* Def. 16. that is, * when the Hyperbola's ſought are equilateral, then the Conſtruction will become much more eaſy. For if CD be drawn perpendicular and equal to CA, and if MPM be drawn through any Point P in the Diameter Aa, parallel to the firſt Diameter Bb; then you need but take in that Line (both ways produced) the Parts PM, PM, &c. each equal to PD, and you will have two Points thro' which the oppoſite Hyperbola's muſt paſs. For becauſe the Triangle PCD, is right-angled at C (ſuppoſing an Hypothenuſe CD to each of them) we ſhall have always $\overline{PD}^2$ or $\overline{PM}^2 = \overline{CP}^2 + \overline{CD}^2$ or $\overline{CA}^2$;
* Art. 127. and therefore the Line PM will be * an Ordinate to the ſecond Diameter Aa, whoſe Conjugate Diameter Bb is equal to it.

DEFINITION.

17.

FIG. 62. Let there be two oppoſite Hyperbola's AM, am, whoſe firſt Axis is the Line Aa, and ſecond the Line Bb; and let there be two other oppoſite Hyperbola's, whoſe firſt Axis is the Line Bb, and ſecond the Line Aa; then the two laſt Hyperbola's BS, bs, are ſaid to be *Conjugates* to the two former ones AM, am; and all the four together are call'd *Conjugate* Hyperbola's.

COROL-

COROLLARY.

132. IT is manifest, that the Lines Ba, Ab, are parallel; because * the Lines Aa, Bb, terminated by them, do bisect each other in the Point C. Whence it follows, by *Def.* 11. that the Hyperbola BS, being the Conjugate to AM, hath CG the Asymptote of the Hyperbola AM for one of its Asymptotes, and Cg the other Asymptote of the same Hyperbola, for the other Asymptote; because those two Lines pass thro' the Centre C, and are parallel to the two right Lines Ba, BA, drawn from (B) the End of the first Axis (Bb) of the Hyperbola BS, to the two Ends A, a, of the second. Therefore it is manifest, that the two right Lines CG, Cg, parallel to Ab, AB, both ways infinitely produced, are not only the Asymptotes of the opposite Hyperbola's AM, am; but likewise are the Asymptotes of the two other Hyperbola's BS, bs, which are Conjugates to them. * *Def.* 4, *and* 5.

PROPOSITION XVI.

Theorem.

133. *IF through any Point* H *in the Asymptote* CG, *which is common to both the Hyperbola's* AM, BS, *there be drawn a Parallel* (MS) *to the other Asymptote* Cg; *I say, that Parallel will meet the said two Hyperbola's in the Points* M, S, *equally distant from the Point* H.

For 1. The Line MS will meet * each of the opposite Hyperbola's AM, BS, in one Point. 2. From the Nature of the Hyperbola AM, the * Rectangle $CH \times HM$ is $= CG \times GA$; and from the Nature of the Hyperbola BS, the Rectangle $CH \times HS$ is $= CG \times GB$. Whence (since * $GB = GA$,) $CH \times HS$ will be $= CH \times HM$; and so HS is $= HM$. *W.W.D.* * *Art.* 104. * *Art.* 101. * *Art.* 88.

COROLLARY I.

134. IF from the Points M, S, in the two Hyperbola's AM, BS, be drawn the Diameters MCm, SCs, terminating in the two other opposite Hyperbola's am, bs; then it is manifest, * that the Diameter Ss will be the second Diameter, which is the Conjugate to the first Diameter (Mm) of the two opposite Hyperbola's AM, am; and contrariwise; the Diameter Mm, will be the second Diameter, which is the Conjugate to the first Diameter Ss, of the opposite Hyperbola's BS, bs. Whence it appears, that any two Conjugate Diameters (Mm, Ss) of the two opposite Hyperbola's AM, am, are * *Art.* 114.

also

alſo two Conjugate Diameters of the two other Hyperbola's *BS*, *bs*, Conjugates to the former; but yet with this Difference, *viz.* that the firſt Diameter *Mm* becomes a ſecond Diameter; and contrariwiſe, the ſecond Diameter *Ss* a firſt.

COROLLARY II.

135. HENCE it is manifeſt, that the Hyperbola's *BS*, *bs*, the Conjugates to the Hyperbola's *AM*, *am*, paſs through the Ends *S*, *s*, of all the ſecond Diameters (*SCs*) of thoſe Hyperbola's: And contrariwiſe, the Hyperbola's *AM*, *am*, paſs through the Ends *M*, *m*, of all the ſecond Diameters (*MCm*) of the two Hyperbola's *BS*, *bs*, which are the Conjugates to them.

The End of the Third Book.

BOOK

BOOK IV.

Of the Three Conick Sections.

DEFINITION.

BY the Term *Conick Section* in general, we understand each of the Curves treated of in the foregoing Books, *viz.* the *Parabola*, the *Ellipsis*, the *Hyperbola*, or the *opposite Hyperbola's*.

PROPOSITION I.

Theorem.

136. *IF through the End* (A) *of any Diameter* (Aa,) *of an Ellipsis, or of any first Diameter of an Hyperbola, the Line* AG *be drawn parallel to the Ordinates* (PM) *thereof, and equal to its Parameter; and if the right Line* aG *be drawn from the other End of that Diameter, cutting any Ordinate* PM *(produced if necessary) in the Point* O. *I say, the Square of the Ordinate* PM, *is equal to the Rectangle under* AP *and* PO. FIG. 63, and 64.

We are to prove, that $\overline{PM}^2 = AP \times PO$.

By the 41st and 55th Articles of the second Book, and the 81st and 118th of the third, we have this Proportion, $Aa : AG :: AP \times Pa : \overline{PM}^2$. But because the Triangles aAG, aPO, are similar; therefore $Aa : AG :: Pa : PO :: AP \times Pa : AP \times PO$. And so $\overline{PM}^2 = AP \times PO$. *W. W. D.*

COROLLARY.

137. HENCE it is evident, that the Square of any Ordinate (PM) to a Diameter Aa, is always less in the Ellipsis, and greater in the Hyperbola, than the Rectangle under the Parameter AG, and the Part (AP) of that Diameter between its End A, and the Point of Concurrence (P) of the Ordinate; whereas * in the Parabola they are equal. And from this Property was it, that *Apollonius*, otherwise

* *Art.* 7, *and* 20.

FIG

wise call'd the great Geometrician, gave those Names, which we have made use of, to the Conick Sections: For by the Word *Parabola* is understood the Exactness or Equality, by the Word *Ellipsis* the Deficiency, and by the Word Hyperbola the Excess, which is found in the Comparison of the Squares of the Ordinates PM, with the Correspondent Rectangles $AP \times AG$.

PROPOSITION II.

Theorem.

FIG. 66, and 67. 138. *In the Ellipsis, every Diameter* Aa, *and in the opposite Hyperbola's, every first Diameter* Aa, *is divided by the Centre* C *into two equal Parts, and meets the Section but in two Points.*

This Proposition has been demonstrated in the 50th Article of the second Book; and in the 96th and 103d Articles of the third Book.

PROPOSITION III.

Theorem.

139. *There can be but one Tangent* (LAL) *passing through a given Point* (A) *in a Conick Section.*

This Proposition is found demonstrated in the 21st Article of the first Book, the 56th of the second, and the 107th of the third Book.

PROPOSITION IV.

Theorem.

140. *The Tangents* LAL, lal, *passing through the Ends* A, a, *of any Diameter of an Ellipsis, or two opposite Hyperbola's, are parallel to each other.*

This has been prov'd in the 44th and 45th Articles of the second Book, and the 110th of the third.

PROPOSITION V.

Theorem.

141. *Any Diameter of an Ellipsis, or the opposite Hyperbola's being given: I say, the Position of the Diameter, which is the Conjugate to it, is so determin'd, as that there can be but one only.*

For 1. If the Section be an Ellipsis, or the opposite Hyperbola's; and if Aa be a first Diameter; then it is manifest, by the 56th

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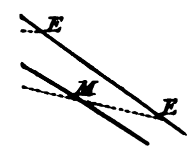
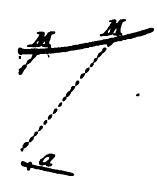




61

56th Article of the second Book, and the 13th Definition of the third Book, that the Conjugate Diameter (Bb) to Aa will be parallel to the Tangent LAL, passing thro' the Point A. Therefore, * *&c.* *Art. 139.

2. When Bb is a second Diameter of two opposite Hyperbola's: This has been demonstrated in the 115th Article of the third Book.

COROLLARY.

142. HENCE it appears, that a Conick Section being given, together with one of its Diameters, the Position of the Ordinates to that Diameter is so determined, that every of them can have but one Diameter, and all of them are parallel to one another. For in the Parabola, the Ordinates must be * parallel to the Tangent * Art. 21. passing through the Origin of the given Diameter, and in the other * * Def. 12, Sections parallel to that Diameter, which is the Conjugate to the II. Diameter given. and 14, III.

PROPOSITION VI.

Theorem.

143. *IN an Ellipsis, every Diameter* Aa, *as also in the opposite Sections, every first Diameter* Aa, *divides the Section into two Parts or Portions,* AM, am, *which being taken in contrary Positions on each Side that Diameter, are similar and equal to one another.*

For take CP, Cp, equal to one another, in the Diameter Aa (produced in the opposite Sections) on each Side the Centre C; and draw the Ordinates PM, Pm; then it is manifest, * that those Ordinates are equal to one another, and the Angles * CPM, Cpm, are equal. Therefore, if the Plane Cpm be suppos'd to be laid upon the Plane CPM on the other Side of the Diameter Aa in a contrary Position, so that the right Line Cp coincides with CP, and pm with PM; then it is manifest, that the Point a will * coincide with A, and the Point m with the Point M. And since this always happens, let the equal Parts CP, Cp be of what Magnitude soever: It follows, that all the Points (m) of the Part am will coincide with all the Points (M) of the Part AM; and consequently the said two Parts or Portions will coincide and be equal. *W. W. D.* * Art. 45, 55, 85, and 118. *Art. 142. *Art. 138.

PROPOSITION VII.

Theorem.

144. *IF the right Line* MPM, *be drawn through any Point* P, *taken in* Aa, *the Diameter of a Conick Section (produced in the Hyperbola, when the same is a principal Diameter) parallel to the Ordinates to* FIG. 68, 69, 70, 71.

 that

that Diameter: I say, the Line MPM *will meet the Section in only two Points* M, M, *equally distant each way from* P. *And contrariwise, if any right Line* MM, *terminating in a Conick Section, be cut into two equal Parts by the Diameter* A a *in the Point* P, *not being the Centre; then the said Line* M M *will be parallel to the Ordinates to that Diameter.*

This has been demonstrated in the 9th, 11th, and 20th Articles of the first Book; the 43d, 45th, and 55th of the second Book; and the 83d, 85th, and 118th of the third.

COROLLARY.

145. HENCE if any right Line *MM*, terminating in a Conick Section, be cut into two equal Parts by the Diameter *A a*, in the Point *P*, not being the Centre; then all the Parallels to that Line terminating in the Section, will be bisected likewise by that Diameter.

PROPOSITION VIII.

Problem.

146. *A Conick Section being given, to find a Diameter thereof.*

Draw the right Lines *M M*, *N N*, parallel to one another, and terminating in the Section; then if they be bisected in the Points *P*, *Q*, the right Line *Aa* drawn thro' *P*, *Q*, will be a Diameter.

* Art. 145. For * the Diameter passing through *P* the Middle of *M M*, must pass likewise through *Q* the middle of *N N*.

COROLLARY. I.

147. IF any other Diameter *D d* be drawn, as above; then it is manifest, that the Conick Section will be * a Parabola, when *D d* is parallel to *A a*; an Ellipsis, * when *D d* meets *A a* within the Section; and lastly, an Hyperbola, * or the opposite Sections, when the Lines *D d*, *A a*, meet each other in the Point *C*, without the Section; in which two last Cases, the Point of Concurrence *C* is the Centre. And so you have here a Succession of the Definitions of the Diameters of the Parabola, Ellipsis, and Hyperbola.

* Def. 7. I. * Def. 9. II. * Def. 9. III.

When the whole Ellipsis is given, you need only draw a Diameter *A a*, for having the Centre thereof: For because the Magnitude of the Diameter is * determin'd by the Ellipsis, you need only bisect that Diameter in *C*. The same is to be done * when opposite Hyperbola's are given.

* Art. 50. * Art. 96.

COROL-

COROLLARY. II.

148. HENCE, a Conick Section being given, together with some Point O, in the same Plane; we may always draw some Diameter Dd, through that Point: For in the Parabola, you need only draw Dd through the given Point O, parallel to any Diameter Aa; and in the Ellipsis, Hyperbola, or opposite Sections, the right Line Dd, passing through the given Point O, and the Centre C, found by the last Corollary.

COROLLARY. III.

149. HENCE it is evident, that a right Line MM can meet a Conick Section in only two Points M, M. For if a Diameter Aa be drawn through P, the Middle of MM; it is manifest, by *Art.* 144. that the same will be parallel to the Ordinates to that Diameter; and so by the same Article, the Line MM can meet the Section in only two Points M, M.

Note, If the Line should pass through the Centre C, then Recourse must be had to *Art.* 138. where this has been prov'd already.

COROLLARY IV.

150. ANY Ellipsis or Hyperbola being given, to find two Conjugate Diameters (Aa, Bb) thereof; and moreover, to draw the Asymptotes CG, Cg, when the given Section is an Hyperbola. FIG. 69, 70.

Find a Diameter Aa, by means of the Parallels MM, NN, and through C draw Bb parallel to those two Lines: Then it is evident, * that Aa, Bb will be Conjugate Diameters; because the Lines MM, NN being divided by the Diameter Aa into two equal Parts in the Points P, Q, will be * Ordinates both ways to that Diameter. * *Def.* 12, II. *and* 14, III. *Art.* 144.

But now to draw (*Fig.* 70.) the Asymptotes CG, Cg; make as $AP \times Pa : \overline{PM}^2 :: \overline{CA}^2 : \overline{CB}^2$ or $\overline{Cb}^2$. or (which is the same thing) make as a mean Proportional between AP and Pa to PM, so is CA to CB or Cb. This being done, if the right Lines AB, Ab are drawn, then the indefinite right Lines Cg, CG, drawn thro' the Centre C parallel to AB, Ab, will be the Asymptotes sought. For it is manifest, that Bb will be * the Conjugate Diameter to the first Diameter Aa; and what remains, is manifest by *Def.* 13 and 14. *Book* 3. * *Art.* 81, *and* 118.

PROPOSITION IX.

Problem.

151. *ANY Conick Section being given, together with one of its Diameters* Aa; *to find the Position of the Ordinates* (PM) *to that Diameter.* FIG. 68, 69, 70, 71.

 Draw

Draw two Parallels to the given Diameter Aa, equally distant each way from the same, and meeting the Section in the Points M, M; then I say, the Line MM, which cuts the given Diameter in the Point P, is an Ordinate both ways to the Diameter Aa, if the Point P does not fall in the Centre.

For, by Construction, the Line MM will be bisected by the Diameter Aa in the Point P; and consequently the Line MM will be * an Ordinate both ways to that Diameter.

*Art. 144.

After this manner may be found always the Position of any Ordinate (PM) to the given Diameter Aa. For 1. in the Parabola, and Hyperbola (*Fig.* 68 and 70.) when the given Diameter Aa is a first Diameter, it is manifest, that at whatever Distance the two Parallels be drawn from the Diameter Aa, they will each meet the Section in one Point M; because the Section * infinitely recedes more and more from the Diameter Aa. 2. In the Ellipsis (*Fig.* 69.) and the opposite Sections (*Fig.* 71.) when the given Diameter Aa is a second Diameter; then it is manifest, that there can be drawn two Parallels on both Sides the Diameter Aa, each of which will cut the Section in one Point M, so that the Line MM will meet the given Diameter Aa in the Point P, not being the Centre; because in the Ellipsis the Ordinates to the Diameter Aa do continually * diminish from the Centre C to A; and contrariwise, in the opposite Sections, do * increase as they go from the Centre C.

* Art. 10, 20, 84, and 118.
* Art. 44, and 55.
* Art. 84, and 118.

COROLLARY I.

FIG. 68, 69, 70.

152. FROM hence arises a new Way of drawing a Tangent through a Point (A) given in a Conick Section. For draw * a Diameter (Aa) through that Point, and find a double Ordinate (MPM) to that Diameter. This being done, it is manifest, * that a Line drawn through the Point A parallel to MM, will touch the Section in A.

*Art. 148.
* Art. 10, 20, 44, 55, 84, 118, and Def. 9, I. 12, II. 7, III.

COROLLARY II.

FIG. 69, 70, 71.

153. HENCE it farther appears, when an Ellipsis, or two opposite Sections are given, together with any one of their Diameters Aa, how to find the Conjugate Diameter (Bb) to that. For you need only draw Bb through the Centre C, parallel to the Ordinates to Aa.

Or else, if Bb be the given Diameter, and the Conjugate to it (Aa) is requir'd, draw MM parallel to Bb, and terminating in the Section; and then through the Points P, C, the Middles of MM and Bb, draw the Diameter Aa sought.

COROL-

COROLLARY III.

154. AN Hyperbola MAM being given, together with one of the second Diameters Bb in Position, to determine the Magnitude thereof, and also to find the Position of the Ordinates to the same. FIG. 70.

First seek the first Diameter Aa, to which Bb is the Conjugate, by the second Part of the last Corollary; and then make $AP \times Pa$: $\overline{PM}^2 :: \overline{CA}^2 : \overline{CB}^2$ or $\overline{Cb}^2$. This being done, it is * manifest, that Bb will be the Magnitude of the second Diameter Bb, and the Ordinates thereof will be parallel to Aa. * *Art.* 81. *and* 118.

PROPOSITION X.

Problem.

155. *FROM a given Point* T *without a given Conick Section, to draw two Tangents* TM, TM, *to that Section.* FIG. 72, 73, 74.

For the Parabola.

Draw * a Diameter through the given Point TC (*Fig.* 72.) meeting the Parabola in A, and take AP equal to AT; moreover, draw * a Parallel to the Ordinates to that Diameter through the Point P, which will meet * the Parabola in the two Points M, M; then if the right Lines TM, TM, be drawn through the given Point T, the said Lines will be * the Tangents sought. * *Art.* 148. * *Art.* 151. * *Art.* 144. * *Art.* 22, *and* 23.

For the Ellipsis.

Draw * the Diameter Aa through the given Point T, and take CP a third Proportional to CT, CA; likewise through P draw a Parallel to the Ordinates, which will meet * the Ellipsis in the Points M, M; then if the right Lines TM, TM, be drawn from the given Point T, those Lines will touch * the Ellipsis. FIG. 73. * *Art.* 148. * *Art.* 144. * *Art.* 57, *and* 58.

For the Hyperbola and opposite Sections.

Draw * the Diameter Aa through the given Point T, and determine * the Magnitude thereof, if it be a second Diameter, and take CP a third Proportional to CT, CA *viz.* on the same Side the given Point T, with regard to the Centre, when that Point falls in one of the Angles formed by the Asymptotes; and on the opposite Side, when it falls in the adjoining Angles: This being done, through the Point P draw a Parallel to the Ordinates, which will meet * the Hy- FIG. 74. * *Art.* 148. * *Art.* 154. * *Art.* 144.

4 perbola,

perbola, or opposite Sections, in the Points M, M; then if the right
*Art. 121. Lines TM, TM be drawn through the Point T, they will be * the Tangents sought.

If the given Point should fall in the Centre C, then would the
*Art. 108. two * Tangents be the Asymptotes CG, Cg; which might be drawn by *Art.* 150. And lastly, if the given Point should fall in one Asymptote, as in S; then there must be drawn through the Point H, the Middle of CS, the Parallel HM to the other Asymptote CG,
*Art. 104. which will * meet the Hyperbola in the Point M, through which, and the given Point S, if the right Line SM be drawn; this Line
*Art. 107. will be * one of the Tangents sought, and the other Tangent will be the Asymptote Cg it self, wherein the given Point S is given.

COROLLARY I.

156. BEcause the Line MPM, being parallel to the Ordinates, al-
*Art. 144. ways meets * the Section in two Points M, M, equally distant each way from the Point P, and not more; it follows, that there can be drawn but two right Lines TM, TM, from a given Point T, without a Conick Section, to touch the same. Whence it is evident, that the Diameter drawn through T, the Point of Concurrence of the Tangents, bisects the Line MM, joining the Points of Contact, in the Point P; and contrariwise, the Diameter bisecting the right Line MM, joining the Points of Contact of the two Tangents MT, MT, in the Point P, passes through T the Point of Concurrence of the Tangents.

COROLLARY II.

157. ANY two Tangents to the Parabola (*Fig.* 72.) will meet each other, being produced if necessary. For if any two Points of Contact M, M, be joined by a right Line; and then if that Line be bisected in the Point P, and the Line AP be taken in the Diameter passing through the Point P, and meeting the Parabola in A, equal to AP; it is manifest, that the two Tangents MT, MT, passing through the Points M, M, will meet one another in the said Point T.

COROLLARY III.

158. IT is farther evident, (*Fig.* 74.) that any two Tangents to an Hyperbola, will meet each other, if produced according to Necessity; and that always within the Angle form'd by the Asymptotes: For if any two Points of Contact M, M, be joined by a straight Line, and after having bisected the same in P, there be taken CT in the

the Diameter passing through that Point (and meeting the Hyperbola in *A*,) a third Proportional to *CP*, *CA*; it is then manifest, that the Tangents *MT*, *MT* will meet each other in the Point *T*, which will be always * within the Angle form'd by the Asymptotes; because the *Art.* 103. Semi-diameter *CA* falls within that Angle.

COROLLARY IV.

159. ANY two Tangents to an Ellipsis, or the opposite Sections, (*Fig.* 73, 74.) will meet each other, when the Line joining the Points of Contact does not pass through the Centre; *viz.* the Tangents to the Ellipsis, on the same Side the Centre as that Line falls, and those to the opposite Sections, on the contrary Side. This is prov'd by means of the Proposition above, as we have shewn in the two last Corollaries.

PROPOSITION XI.

Problem.

160. *A Conick Section being given, to find a Diameter thereof, which shall make Angles either way with the Ordinates equal to an Angle given.*

For the Parabola.

Find * *AP* one of its Diameters, and draw the Line *AN* through *A* * *Art.* 146. the Origin thereof, making the Angle *PAN* with *AP*, equal to the FIG. 75, given Angle, and meeting the Parabola in the Point *N*; then bisect 76. *AN* in *O*, and draw *OM* parallel to *AP*; and, I say, the Line *OM* is the Diameter sought.

For 1. Because all the Diameters of a Parabola must be parallel to one another, by *Def.* 7. *Book* I. therefore the Line *MO* will be a Diameter; since *AP* is one likewise.

2. Because the Line *AN*, terminating in the Parabola, is bisected by the Diameter *MO*; therefore that Line will be * an Ordinate * *Art.* 144. each way to the Diameter *MO*.

3. Because *MO*, *AP*, are parallel, the Angle *MOA*, made by the Diameter *MO*, and its Ordinate *OA*, will be equal to the Angle *PAN*, which was made equal to the given Angle. Therefore, &c.

If the Angle given be a right one, then it is manifest, that the Diameter *MO*, found as above, will be * the Axis of the Parabola. * *Art.* 23.

For the other Sections.

*Art. 146. Fig. 77, 78, 79, 80. Find * Aa one of the Diameters, and upon the same describe the Segment ANa of a Circle, that may contain an Angle equal to the Angle given, or the Complement thereof to two right Angles; then draw the right Lines NA, Na, from the Point N, wherein the Circle cuts the Section, to the Ends A, a, of the Diameter Aa. This being done, if through the Points O, Q, being the Middles of NA, Na, and the Centre C, you draw the two Diameters Mm, Ss; I say, each of these Diameters make Angles with their Ordinates equal to the given Angle.

*Art. 144. For because the Line AN, terminating in the Section, is bisected by the Diameter Mm in the Point O, the said Line AN will be * an Ordinate both ways to that Diameter; since it bisects the Lines Aa, AN in the Points C, O: Therefore the Angle mOA made by the Diameter Mm and Ordinate AO, will be equal to the Angle aNA, which by Construction is equal to the Angle given, or its Complement to two right Angles. After the same manner we prove, that the Diameter Ss makes an Angle with its Ordinate QN, equal to the given Angle, or its Complement to two right ones. Whence, &c.

* Def. 12, II. and 14, III. *Art. 58, and 128. It is manifest, 1. That the Diameter Ss is * the Conjugate to the Diameter Mm; because the same is parallel to the Ordinate ON. 2. The Conjugate Diameters Mm, Ss will become * the two Axes, when the Angle given is a right Angle.

PROPOSITION XII.

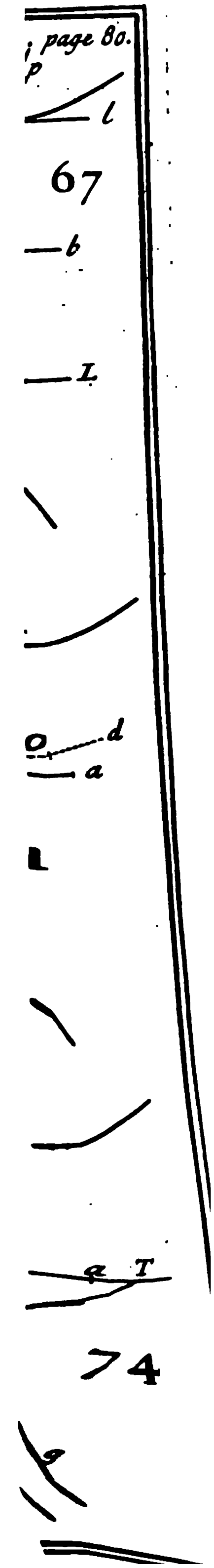
Problem.

161. *ANY Diameter of a Conick Section being given, together with the Parameter thereof, and the Position of its Ordinates; and knowing moreover, whether the same be a first or second Diameter, when we have to do with the Hyerbola, to describe all the three Sections by one uniform Method.*

First Way.

* Art. 27. Fig. 81. *For the Parabola.* Find * the Axis AP, its Origin A, and Parameter AG, assum'd in the Axis continued out beyond A, and draw an indefinite right Line DD through the Point G perpendicular to PG. This being done, if the indefinite right Line DM be mov'd along DG, always parallel to AG; so that the End D thereof carries along with it the Side DA, of the right Angle DAM, whose Vertex A is moveable about A the Origin of the Axis: I say, the continual Intersection

page 80.



tersection (M) of the Line DM, and the Side AM, will by this Motion describe the Parabola sought.

For if MP be drawn perpendicular to the Axis, then the right-angled Triangles AGD, MPA, will be similar; because each of the Angles GAD, PMA, being added to the Angle PAM, makes up a right Angle. Therefore we have this Proportion, *viz.* $AG : GD$, or $PM :: PM : AP$. And so $\overline{PM}^2$ will be $= GA \times AP$; and consequently PM will be * an Ordinate to the Axis AP. * Art. 7.

Note, This Construction has been already laid down in Book I. Art. 29. so as to agree to all Diameters: And the Reason why we have here only confin'd the same to the Axis, is to show its Affinity with the Construction we are going to give for the other Sections.

For the other Sections. Find a mean Proportional between the given Diameter and its Parameter, and place the same so that it be parallel to the Ordinates, and divided into two equal Parts by the Centre; then it is manifest, * that we shall have two Conjugate Diameters; * Def. 13, by means of which find * the two Axes, and the Parameter of either II. and of them in the Ellipsis, and that of the first Axis in the Hyperbola. 15, III. This being done, * Art. 64,

Produce the Axis of the Ellipsis to G, and divide the Axis (Aa) of and 128. the Hyperbola in G; so that aG be to GA, as the Axis Aa is to its Fig. 82, Parameter; and draw the indefinite right Line DD through the Point 83. G perpendicular to Aa; then if the Point D moves along this Line, and at the same time this Point carries with it the right Line Da, moveable about the End a of the Axis (Aa); as also the Side DA, of the right Angle DAM, moveable about its Vertex A, plac'd on the other End (A) of the said Axis Aa: I say, the continual Intersection (M) of the right Lines AM, aD, will by this Motion describe the Section requir'd.

For if MP be drawn perpendicular to the Axis Aa, the similar Triangles aPM, aGD, will give this Proportion, *viz.* $aP : PM :: aG : GD$. But the right-angled Triangles AGD, MPA, are similar; because each of the Angles GAD, PMA, being added to the Angle PAM, makes a right Angle: And therefore $AP : PM :: GD : GA$. And by multiplying the Antecedents and Consequents of the two first Ratio's, by the Antecedents and Consequents of the two last, there will arise $aP \times PA : \overline{PM}^2 :: aG \times GD : GD \times GA :: aG : GA$; that is, as the Axis Aa to its Parameter. Therefore, * &c. * Art. 41,

Note, The farther the Point D is distant from G, the greater will and 81. the Angle PaM be; and contrariwise, the less the Angle PAM; so that when the Lines aM, AM, becoming parallel in the Hyperbola, and afterwards cutting one another on the other Side the Line DD,

their continual Interſection will deſcribe the oppoſite Hyperbola.

In the Ellipſis and Hyperbola, if the Point a be conceiv'd to be infinitely diſtant from A, or (which is the ſame thing) if the Axis Aa be ſuppos'd infinitely great; then the Lines GA, Da, which will meet one another at an infinite Diſtance, may be taken for Parallels. And ſo this laſt Conſtruction falls again into the Caſe of the foregoing Conſtruction. Therefore the Ellipſis or Hyperbola will then become a Parabola, whoſe Parameter is the Line AG; and conſequently a Parabola may be eſteem'd as an Ellipſis or Hyperbola, whoſe Axis is infinite, *viz.* the firſt Axis in the Hyperbola, and either of the two in the Ellipſis.

Second Way.

Fig. 84. *For the Parabola.* Let HAL be an Iſoſcelles Triangle, one of whoſe Sides AH is ſituate in the given Diameter AP indefinitely produced both ways from its Origin A, and the other Side AL in the indefinite Tangent LAL paſſing through the Point A. Now if HL, the Baſe of this Triangle, be ſuppos'd to move always parallel to it ſelf; ſo that one End L thereof carries along with it the indefinite right Line LM parallel to AP, and the other End H, the Line FH, parallel to AL, and equal to the given Parameter of the Diameter AP; and if the Extremity F of the Line FH, likewiſe carries along with it the right Line FA, moveable about the fix'd Point A: Then, I ſay, the continual Interſection (M) of the two right Lines FA, LM, during the Motion of the Line HL in the Angle HAL, and that vertical thereto, will deſcribe the Parabola MAM requir'd.

For if the Ordinate MP be drawn to the Diameter AP, the ſimilar Triangles AHF, APM, will give AH, or AL, or $PM : HF ::$ $AP : PM$, and therefore $\overline{PM}^2 = AP \times HF$. Whence, * &c.

* Art. 7, and 20.

It muſt be obſerv'd here, that the Point H falls beyond A, the Origin of the Diameter AP, when the Points F, L fall on both Sides that Diameter.

Fig. 85, 86. *For the other Sections.* Here the Conſtruction is the ſame as that of the Parabola, only the Line LM muſt move about a, the other End of the given Diameter Aa; whereas in the Parabola, LM muſt be parallel to Aa. *Note*, In the Hyperbola, we ſuppoſe that the given Diameter is a firſt Diameter; for if it be a ſecond Diameter, the firſt Diameter being the Conjugate to it may be found, as alſo its Parameter by *Art.* 115.

For if the Ordinate MP be drawn to the Dinmeter Aa; then the ſimilar Triangles aPM, aAL, and APM, AHF, will give theſe Proportio s, $aP : PM :: aA : AL$ or AH. And $AP : PM :: AH :$ HF.

HF. And therefore, if the Antecedents and Consequents of the two first Ratio's be multiply'd by those of the second, there will arise $aP \times PA : \overline{PM}^2 :: aA \times AH : AH \times HF :: aA : HF$. Whence, * &c. * *Art.* 41, 55, 81, *and* 118.

It must be observ'd, that the Points *H*, *a*, ought to fall on both Sides the Point *A* in the Ellipsis, and on one Side in the Hyperbola, when the Points *F*, *L*, fall on both Sides the Diameter *Aa*.

COROLLARY I.

162. HENCE, if any Diameter *Aa*, together with one of its Ordinates *PM* be given; the Parameter *HF* thereof may be found. For 1. In the Parabola, assume *AH*, in the Diameter *AP*, equal to *PM*; and then if the Line *HF* be drawn parallel to *PM*, and terminated in *F* by the Line *AM*, drawn through *A* the Origin of the Diameter, and *M* the Extremity of the Ordinate: I say, it is manifest, that the Line *HF* will be the Parameter of the Diameter *AP*. FIG. 84.

2. In the other Sections, draw the Line *aM* through *a*, one End of the given Diameter *Aa*, meeting the Tangent *AL*, passing thro' the other End, in the Point *L*; then assume *AH*, in the Diameter *Aa*, equal to *AL*, and draw *HF* parallel to *PM*, meeting the Line *AM* in *F*, and the Line *HF* will be the Parameter of the Diameter *Aa*. FIG. 85, 86.

COROLLARY II.

163. HENCE arises a very exact and uniform practical Method for describing a Conick Section thro' several Points: An Example of which I shall give in the Ellipsis, which will serve as a Rule for describing the other Sections.

Assume *AG*, in the Tangent *AL*, passing through (*A*) one End of the given Diameter *Aa*, equal to the Parameter of that Diameter, and draw the indefinite right Line *GF* parallel to *Aa*; and through the Point *A* draw any Number of right Lines *AF*, *AF*, &c. at pleasure. This being done, assume *AL*, *AL*, &c. in the indefinite Tangent each equal to its Correspondent *GF*, *GF*, &c. and draw the right Lines *aL*, *aL*, &c. then, I say, the Intersections *M*, *M*, &c. of the correspondent right Lines *FA*, *La*, *FA*, *La*, &c. will be Points of the Ellipsis, whose Diameter is the Line *Aa*, Tangent the Line *AL*, and the Parameter of the Diameter *Aa*, the Line *AG*. This is manifest by drawing *FH* parallel to *AG*, and also the Line *HL* through the Point *L*, answering to *F*. For the Triangle *HAL* will be Isoscelles, because * *AL* is equal to *GF*, or *AH*, and *HF* will be the FIG. 87. * *Hyp.*

the Parameter of the Diameter Aa; and so this Construction falls again into that of the second of the two abovesaid ways.

Because the Lines GF, AL, become very great, when the Points (M) to be found are near to the Point a; these Points may be found, by using the Tangent al passing through the other End a of the Diameter Aa, together with the Line gf Parallel to Aa, as may be seen in the Figure.

If the Ordinates MP, MP, &c. be drawn parallel to the Tangent AL, and if they be produced on the other Side of the Diameter Aa, to the Points M, M, &c. so that each of them be cut into two equal Parts by that Diameter; then it is * evident, that the new Points M, M, &c. will be moreover in the same Ellipsis.

* Art. 43.

The same Extent of your Compasses, *viz.* GF, or AL, will serve for denoting any Number of Points F, F, &c. L, L, &c. at pleasure, in the Lines GF, AL. For by that means, all those small Parts being equal to one another, every GF will be equal to its Correspondent AL: Which is the Foundation of the Demonstration.

PROPOSITION XIII.

Theorem.

FIG. 88, 89, 90, 91. 264. *IF there be two right Lines* MN, AR, *terminating in a Conick Section, meeting each other in* P, *and if they be parallel to two right Lines given in Position: I say, the Rectangle* MP × PN *will be always to the Rectangle* AP × PR, *in a given Ratio, in whatsoever Part of the Section the said right Lines* MN, AR *fall.*

For the Parabola.

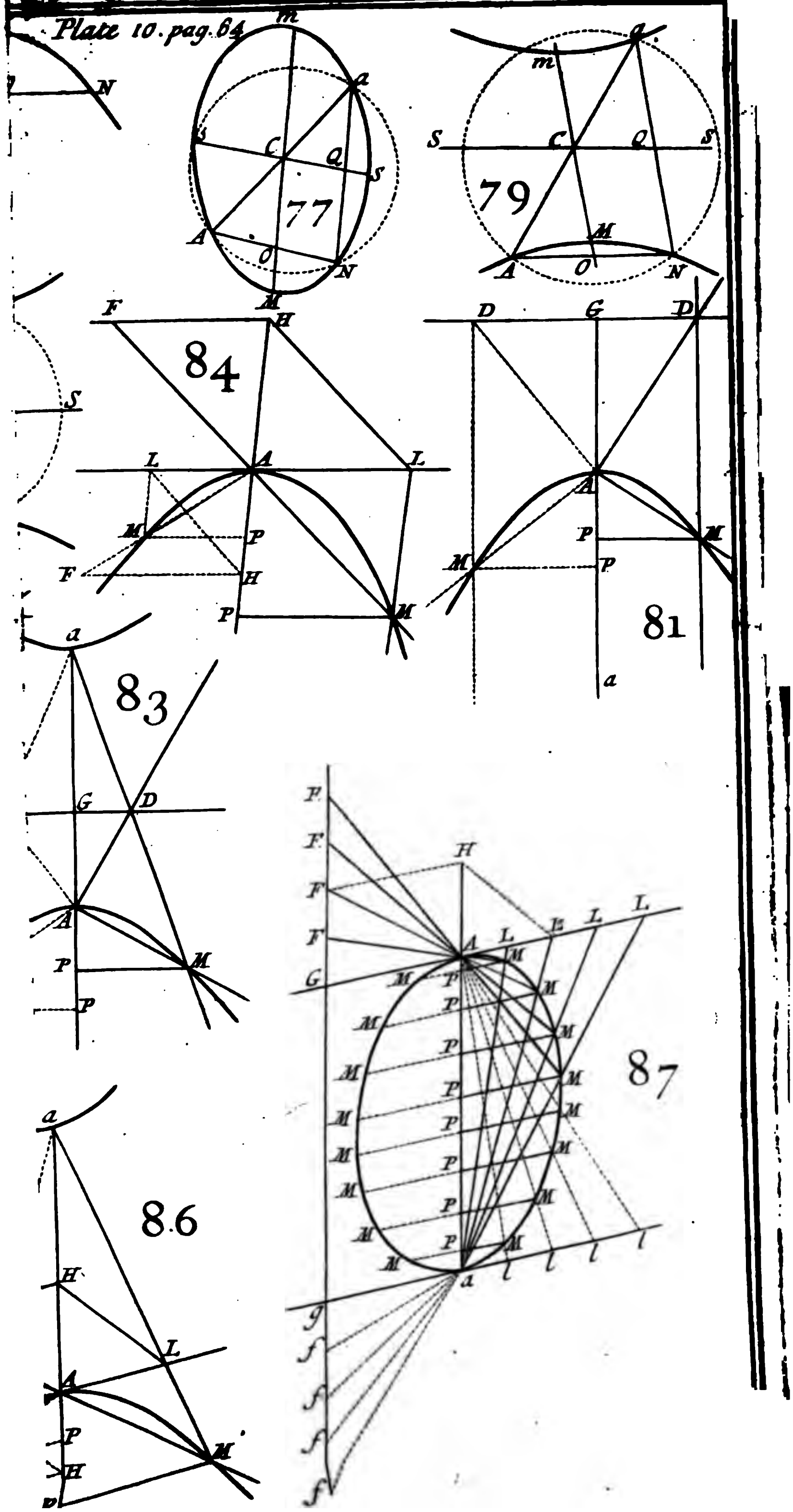
FIG. 88. Let the Tangents CB, EB, meeting one another in the Point B, be parallel to the right Lines MN, AR: I say, $MP \times PN : AP \times PR :: \overline{CB}^2 : \overline{EB}^2$.

For if the Diameter CG be drawn * through G, the Middle of MN; and if CB be drawn through C, the Origin thereof, parallel to MN; then it is * manifest, that CB will touch the Section in C. And after the same manner must the Tangent EB be drawn parallel to AR, which produce till it meets the Diameter CG in the Point K; and if the Ordinate EL be drawn through the Point of Contact E, we shall have * $KC = CL$; and so $KB = BE$. Again, draw the Ordinate AD, and the Line AE parallel to the Diameter CG, and call the given Quantities KB or BE, m; BC, n; CK, e; the Parameter (CH) of the Diameter CG, p; and the indeterminate Quantities AP, x; PM, y; AD, r; CD, s.

* Art. 148.
* Art. 10, and 21.
* Art. 22, and 23.

This

Plate 10. pag. 64



This being laid down, because the Triangles KBC, APF, are similar, therefore PF is $=\frac{nx}{m}$, AF or $DG=\frac{ex}{m}$: And consequently $CG=\frac{ex}{m}+s$, GM or $GN=y+\frac{nx}{m}+r$, PN or $GN+GP=y+\frac{2nx}{m}+2r$; $MP\times PN=yy+\frac{2nx}{m}y+2ry$, $\overline{GM}^2=yy+\frac{2nx}{m}y+2ry+\frac{nn}{mm}xx+\frac{2nr}{m}x+rr$. But * $CD(s):CG\left(\frac{ex}{m}+s\right)$ :: $\overline{AD}^2(rr):\overline{GM}^2=rr+\frac{err}{ms}x=rr+\frac{ep}{m}x$, because $\overline{AD}^2(rr)=CD\times CH(ps)$. And comparing the two Values of $\overline{GM}^2$ together, there will be form'd the following Equation $yy+\frac{2nx}{m}y+2ry+\frac{nn}{mm}xx+\frac{2nr}{m}x-\frac{ep}{m}x=0$, which agrees to all Points of the Parabola, when the Line AR falls above the Diameter CG, and the Point of Intersection P falls between the Points A, R. * Art. 8, and 20.

Now in the aforesaid Equation if you make $y=0$, and strike out all the Terms affected with y, we shall have $\frac{nn}{mm}xx+\frac{2nr}{m}x-\frac{ep}{m}x=0$. From hence we get $x=\frac{emp}{nn}-\frac{2mr}{n}=AR$; because when $PM(y)$ is equal to nothing it is manifest that $AP(x)$ will be equal to AR. Therefore $AP\times PR=\frac{emp}{nn}x-\frac{2mr}{n}x-xx$; and consequently $MP\times PN$ $\left(yy+\frac{2nx}{m}y+2ry\right)$: $AP\times PR\left(\frac{emp}{nn}x-\frac{2mr}{n}x-xx\right)$ :: $\overline{CB}^2(nn)$: $\overline{EB}^2(mm)$: because by multiplying the Means and Extremes, the preceeding Equation will come out again. And since the Tangents CB, BE remain always the same, let their Parallels MN, AR, fall in any Part of the Parabola; therefore, &c.

Here it must be observ'd that there may happen different Cases, according to the different Positions of the right Lines MN, AR; but since the Demonstration is every where the same, excepting only in the Alteration of some Lines, or Terms that vanish; therefore I shall not explain all the Cases separately; which I would have observ'd also in the other Sections.

For the other Sections.

If the two Semi-diameters CO, CB, be drawn Parallel to the right Lines MN, AR: I say $MP\times PN:AP\times PK$ :: $\overline{CO}^2:\overline{CB}^2$. FIG. 89, 90, 91.

Draw

Draw the Diameter CG, having the Line MN for a double Ordinate, and draw BE, AD, parallel to MN; alſo draw AF parallel to CG, and call the given Quantities CB, m; BE, n; CE, e; and the Semi-diameter CK, t; the Semi-conjugate to it CO, c; and the intermediate Quantities AP, x; PM, y; AD, r; CD, s.

This being premiſed, becauſe the Triangles CBE, APF are ſimilar, therefore PF is $=\frac{nx}{m}$, AF or $DG=\frac{ex}{m}$. And conſequently in the Hyperbola or oppoſite Sections, (*Fig*, 90. and 91.) we have $CG=\frac{ex}{m}\pm s$, GM or $GN=y+\frac{nx}{m}-r$, PN or $GN+GP=y+\frac{2nx}{m}-2r$; $MP\times PN=yy+\frac{2nx}{m}y-2ry$, $\overline{GM}^2=yy+\frac{2nx}{m}y-2ry+$
* *Art.* 82, *and* 118. $\frac{nn}{mm}xx-\frac{2nr}{m}x+rr$. But * $\overline{CD}^2\mp\overline{CK}^2(ss\mp tt):\overline{CG}^2\mp\overline{CK}^2$ $\left(\frac{eexx}{mm}\pm\frac{2esx}{m}+ss\mp tt\right)::\overline{AD}^2(rr):\overline{GM}^2=rr+\frac{eerrxx\pm 2emrrsx}{mmss\mp mmtt}$
* *Art.* 82, *and* 118. $=rr+\frac{eeccxx\pm 2eccmsx}{mmtt}$, by ſubſtituting $\frac{cc}{tt}$ for $\frac{rr}{ss\mp tt}$ the Value * thereof. And comparing the ſaid two Values of $\overline{GM}^2$, we ſhall form the following Equation $yy+\frac{2nx}{m}y-2ry+\frac{nntt-ccee}{mmtt}xx-\frac{2nrtt\mp 2cces}{mtt}x=o$; in which, if you ſubſtitute $cctt$ in the Place of its Value $nntt-ccee$ (where note when CB is half of a ſecond
*Art.134. Diameter, you muſt ſuppoſe * a Conjugate Hyperbola to paſs through
* *Art.* 81, *and* 118. the Point B;) becauſe * $\overline{CE}^2+\overline{CK}^2(ee+tt):\overline{EB}^2(nn)::\overline{CK}^2(tt):\overline{CO}^2(cc.)$ then there will ariſe $yy+\frac{2nx}{m}y-2ry+\frac{cc}{mm}xx-\frac{2nrtt\mp 2cces}{mtt}x=o$, which Equation agrees to all Points of the Section, when the Points A, R, fall on both Sides the Diameter CG, and the Point of Interſection P falls between the Points A, R.

Now in the ſaid Equation, if you make $y=o$, there will ariſe (by ſtriking out all the Terms affected with y) $\frac{cc}{mm}xx-\frac{2nrtt\mp 2cces}{mtt}x=o$, and ſo we ſhall get $x=\frac{2mnrtt\mp 2ccems}{cctt}=AR$; becauſe when PM (y) is equal to nothing, then AP (x) will be $=AR$. Therefore $AP\times PR$ $\left(\frac{2mnrtt\mp 2mcces}{cctt}x-xx\right):MP\times PN\left(yy+\frac{2nx}{m}y-2ry\right)::\overline{CB}^2(mm):\overline{CO}^2(cc)$. For if the Means and Extremes of this Proportion be multiply'd, there will ariſe the preceding Equation. Now

because the Semidiameters CO, CB remain always the same, let their Parallels MN, AR fall in any Part of the Section: Therefore, *&c.*

I shall not here lay down the Calculus for the Ellipsis in particular, because it differs only in some Lines from that of the Hyperbola.

COROLLARY I.

165. IF any two right Lines MN, AR, terminating in a Conick Section, meet each other in P; and if any two other right Lines FG, BD be drawn parallel to the two former ones, terminating in the Section likewise, and meeting one another in the Point Q; then it is manifest, that $MP \times PN : AP \times PR :: FQ \times QG : BQ \times QD$. For the two right Lines AR, BD, being parallel to one another, will be parallel to the same right Line CZ given in Position; as likewise the two right Lines MN, FG, to the same right Line CT, also given in Position. FIG. 92.

COROLLARY. II.

166. IF there be two Parallels AR, BD, terminating in a Conick Section, and meeting any right Line FG, terminated by the same Section in the Points E and Q; then I say, $FE \times EG : AE \times ER :: FQ \times QG : BQ \times QD$. For if in *Corol.* 1. the Line MN be suppos'd to fall in FG, it is manifest, that the Rectangles $MP \times PN$, $AP \times PR$, will become $FE \times EG$, $AE \times ER$.

COROLLARY. III.

For the Circle.

167. FROM this Theorem may be drawn that most noted Property of the Circle, *viz.* that if through any Point P, taken within or without a Circle, there be drawn any Number of right Lines AR, MN, HL, &c. terminated by the Circumference, all the Rectangles $AP \times PR$, $MP \times PN$, $HP \times PL$, &c will be equal to one another: For if the Semidiameters CB, CO, CD, &c. be drawn parallel to these Lines, then (by the Theorem) it is manifest, that all those Rectangles will be to one another, as the Squares of the said Semidiameters or Radii, which by the essential Property of the Circle are all equal to one another. FIG. 93.

COROLLARY IV.

For the Parabola.

FIG. 94. 168. IF there be any right Line MN terminating in a Parabola; and if the Diameter AF be drawn through any Point A in the Parabola, meeting the Line MN in the Point F: I say, the Rectangle $MF \times FN$, is equal to the Rectangle under AF and CH, the Parameter of the Diameter CG, passing through the Middle of MN.

For suppofing, in the Theorem, the Line AP to fall in AF, then it is manifeft, that the Line PF $\left(\frac{n}{m}x\right)$ will become equal to nothing, and so $\frac{n}{m} = o$. Therefore ftriking out all the Terms affected with $\frac{n}{m}$ in the Equation for the Parabola, *viz.* $yy + \frac{2nx}{m}y + 2ry + \frac{nn}{mm}xx + \frac{2nr}{m}x - \frac{ep}{m}x = o$, we fhall have this $yy + 2ry - \frac{ep}{m}x = o$. But $AF = \frac{ex}{m}$, $CH = p$, and $MF \times FN = yy + 2ry$. Whence &c.

I have laid down this Corollary only to fhew the great Extent of the Theorem it is deduced from; for it may be demonftrated more eafy another way, which is thus: $\overline{GM}^2 = GC \times CH$, $\overline{AD}^2$ or $\overline{GF}^2 = DC \times CH$, and therefore $\overline{GM}^2 - \overline{GF}^2$ or $MF \times FN = \overline{GC - DC} \times CH = AF \times CH$.

COROLLARY V.

For the Parabola.

169. HENCE it is manifeft,

1. If there be two right Lines MN, EL, terminating in a Parabola, and parallel to one another; and if through any two Points A, B, in that Parabola be drawn two Diameters AF, BP, meeting the Lines MN, EL, in the Points F, P: I fay it is evident, that $MF \times FN : EP \times PL :: AF : BP$. For the Diameter CG which paffes through the middle of MN, paffes alfo through the middle of EL; and confequently the Rectangle $EP \times PL$ is $= BP \times CH$, and moreover $MF \times FN = AF \times CH$.

2. If any right Line MN terminating in a Parabola, meets two Diameters (AF, BK,) thereof in the Points F, K; then we fhall have this proportion, $MF \times FN : MK \times KN :: AF : BK$.

3. If

3. If any two parallel right Lines MN, EL, terminating in a Parabola, meet any one of its Diameters (BP) in the Points K, P; then will it be as $MK \times KN : EP \times PL :: BK : BP$.

COROLLARY VI.

For the Parabola.

170. HENCE appears the Way of describing a Parabola passing through three given Points A, M, N, and whose Diameters AF, CG shall be parallel to a right Line given in Position; together with the manner of demonstrating, that there can be but one Parabola only that will satisfy the Conditions of the Problem.

For join two of the given Points M, N, by the right Line MN, and through the third A draw a Diameter AF parallel to the Line given in Position, and meeting the Line MN in the Point F; also through (G) the Middle of MN, draw GC parallel to AF. This being done, make $MF \times FN : MG \times GN$, or $\overline{GM}^2 :: AF : GC$. And having taken CH, a third Proportional to CG, GM, describe * a Parabola, with the Parameter CH, and Diameter CG, whose Origin is C, and Ordinates are parallel to MN; and that will be the Parabola requir'd. * *Art.* 29, *and* 30.

For 1. The Parabola will * pass through the Points M, N; because by Construction $CH \times CG = \overline{GM}^2$ or $\overline{GN}^2$. 2. It will pass also through the Point A, because $MG \times GN : MF \times FN :: CG : FA$. 3. The Diameters AF, CG, will be parallel to the right Line given in Position. * *Art.* 7, *and* 20.

And because the right Line CG, whose Origin is C, is a Diameter of the Parabola answering the Conditions of the Problem, and the determinate Line CH is the Parameter of that Diameter; therefore there is but one Parabola only that will satisfy the Problem.

COROLLARY VII.

For the Parabola.

171. IF there be two right Lines AR, MN, terminating in a Parabola, meeting each other in the Point P; and if it be made as $AP \times PR : MP \times PN :: \overline{AP}^2 : \overline{PF}^2$: and the Line AF be drawn; then, I say, the Line AF will be a Diameter. For if the Tangents CB, EB, be drawn parallel to the right Lines MN, AR, and the Diameter CG be drawn through (C) the Point of Contact meeting EB (produced) in K, we shall have $\overline{EB}^2$ or $\overline{KB}^2 : \overline{BC}^2 :: AP \times PR :$ FIG. 88.

$MP \times PN :: \overline{AP}^2 : \overline{PF}^2$, and consequently $KB : CB :: AP : PF$. Therefore the Triangles KBC, APF, will be similar, and their Sides AF, KC parallel: Whence the Line AF, being parallel also to the
* Def. 7, 1. Diameter CG, will be a Diameter; because all Diameters * of the Parabola are parallel to one another.

Corollary VIII.

For the Parabola.

172. By means of the last Corollary, a Parabola may be describ'd, which shall pass through four given Points A, M, R, N.

For if the said four Points be joined by two right Lines AR, MN, intersecting each other in the Point P; and if you make $AP \times PR : MP \times PN :: \overline{AP}^2 : \overline{PF}^2$, and draw the Line AF; then if a
* Art. 170. Parabola be describ'd * passing through the three Points A, M, N, whose Diameters are parallel to the Line AF; this Parabola will be that requir'd. For by the Theorem, the Line AP must meet that Parabola in the Point R, being such that $AP \times PR : MP \times PN :: \overline{EB}^2$ or $\overline{KB}^2 : \overline{BC}^2 :: \overline{AP}^2 : \overline{PF}^2$.

Fig. 95. If the Point F falls beyond the Point P, then there may be another Parabola describ'd, which will pass through the four given Points. But it must be here observ'd, that when one of the Points F falls on one of the given Points M or N; then there will be but one Parabola answering the Problem; and when both of them fall on the Points M, N, the Problem will be impossible; because then the Diameter (AF) of the Parabola, will meet the Parabola in two Points, which has
* Art. 10. been * demonstrated to be impossible.

Corollary IX.

For the Hyperbola, or opposite Sections.

Fig. 96, 97. 173. If there be a right Line MN, terminating in an Hyperbola, or the opposite Sections, meeting an Asymptote CB in the Point Q, and which is parallel to a right Line given in Position; and if thro' any Point A in the Section, there be drawn a right Line AP parallel to that Asymptote, meeting the Line MN in the Point P: I say, the Rectangle $MP \times PN$ to the Rectangle $2\,AP \times PQ$ will be in a given Ratio, let the right Lines MN, AP fall in any Part soever of the Section or Sections.

For if in the Theorem (*Fig.* 90, 91.) the Semidiameter CB be sup-
* Art. 102. posed to become an Asymptote; then it is * manifest, that each of the three Sides of the Triangle CBE will become infinite. And so if KS be drawn (*Fig.* 96, 97.) through K, the End of the Diameter CK, (passing through the Middle of MN) parallel to MN, and meet-

meeting the Asymptote CB in S; then the Triangle CKS will be form'd, whose three Sides are finite, and this will be similar to the Triangle CBE; therefore CK (t) : KS, or * CO (c) : : CE (e) : EB (n); and so ce is $= nt$. Now substituting nt in the Place of its Value ce, in the Equation found in the Theorem for the Hyperbola, *viz.* $yy + \frac{2nx}{m}y - 2ry + \frac{nntt - ccee}{mmtt}xx - \frac{2nrtt \mp 2eccs}{mtt}x = 0$, there arises $yy + \frac{2nx}{m}y - 2ry - \frac{2nrt \mp 2ncs}{mt}x = 0$, or $yy + \frac{2nx}{m}y - 2ry = \frac{2nrt \pm 2ncs}{mt}x$. But producing AD, (if necessary) till it meets the Asymptote CB in H, then the similar Triangles CKS, CDH, will give this Proportion, *viz.* CK (t) : KS (c) : : CD (s) : $DH = \frac{cs}{t}$. And therefore, AH or $PQ = \frac{rt \pm cs}{t}$. Whence we have $MP \times PN\ (yy + \frac{2nx}{m}y - 2ry) : 2AP \times PQ\ (\frac{2rt \pm 2cs}{t}x) : : EB\ (n) : CB\ (m) : : KS : CS$. Because in multiplying the Means and Extremes, the aforesaid Equation will again be produced. Now because the Lines KS, CS, remain always the same, let the Lines MN, AP fall in any Part soever of the Section; since the Diameter LK, passing through the Middle of MN, also * passes thro' the Middle of all the Parallels to MN terminated by the Section, let them fall any how soever. Therefore, &c.

*Art. 113.

*Art. 145.

This Corollary may be immediately demonstrated, without having recourse to the Theorem, after the following manner. Let the given Quantities CK be $= t$, KS or $CO = c$, $CS = m$, and the indeterminate Quantities $CD = s$, AD or $DI = r$, $AP = x$, $PM = y$. Then because the Triangles CSK, APF, are similar, therefore PF is $= \frac{cx}{m}$, AF or $DG = \frac{tx}{m}$; and therefore GM or $GN = y + \frac{cx}{m} - r$, $CG = \frac{tx}{m} + s$. Now because the Triangles CKS, CDH, CGQ, are similar; therefore $CK\ (t) : KS\ (c) : : CD\ (s) : DH = \frac{cs}{t} : : CG\ (\frac{tx}{m} + s) : GQ = \frac{cx}{m} + \frac{cs}{t}$. And therefore $MQ \times QN$, or $\overline{GQ}^2 - \overline{GM}^2 = \frac{2ccsx}{mt} + \frac{ccss}{tt} - yy - \frac{2cxy}{m} + 2ry + \frac{2crx}{m} - rr =$ *$AH \times HI$, or $\overline{DH}^2 - \overline{DI}^2 = \frac{ccss}{tt} - rr$; and by striking out from each Side $\frac{ccss}{tt} - rr$, and bringing over all the Terms wherein y is, to one Side,

FIG. 96.

*Art. 97.

Side, we shall get this Equation, $yy + \frac{2cxy}{m} - 2ry = \frac{2ccsx}{mt} + \frac{2crx}{m}$, which being reduced to a Proportion, gives $MP \times PN\ (yy + \frac{2cxy}{m} - 2ry)$: $2AP \times PQ\ (\frac{2csx}{t} + 2rx) :: KS\ (c) : CS\ (m.)$ *W.W.D.*

The Demonstration is the same, for the opposite Sections only with the Alteration of some Signs.

COROLLARY X.

For the Hyperbola, or opposite Sections.

FIG. 98. 174. IT follows from the last Corollary,

1. If there be two parallel right Lines MN, HG, terminating in an Hyperbola, or the opposite Sections, and meeting one Asymptote CS in the Points Q, I; and if thro' any two Points (A, B) in the Section, there be drawn the two Parallels AP, BD, to the Asymptote CS, meeting the Parallels MN, HG, in the Points P, D; then the Rectangles $MP \times PN$, $2AP \times PQ$, will be to one another, as the Rectangles $HD \times DG$, $2BD \times DI$; and therefore we have $MP \times PN : HD \times DG :: AP \times PQ : BD \times DI$.

2. If there be two parallel right Lines MN, HG, terminating in an Hyperbola, or the opposite Sections, and meeting one Asymptote CS in the Points Q, I; and if, through any Point A in the Section, there be drawn AO parallel to CS, and meeting the Lines MN, HG in the Points P, O; then we shall have the following Proportion, (by conceiving in the last Case the Line BD to fall in AP) $MP \times PN : HO \times OG :: AP \times PQ : AO \times OI :: AP : AO$, because PQ is $= OI$.

3. If there be any right Line HG terminating in an Hyperbola, or the opposite Sections, and meeting the Asymptote CS in the Point I; and if thro' any two Points (A, B) in the Section, there be drawn the Lines AO, BD, parallel to CS, and meeting the Line HG in the Points O, D; then we shall have this Proportion, $HO \times OG : HD \times DG :: AO \times OI : BD \times DI$. This is a farther Continuation of the first Case, in supposing the Line MN to fall in HG.

COROLLARY XI.

FIG. 92. 175. IF any right Line BD, meeting a Conick Section in two Points B, D, be suppos'd to move parallel to itself, until it becomes the Tangent LS; then it is manifest, that the two Points of Intersection B, D will coincide in the Point of Contact L; and so a Point

Point of Contact may be consider'd as two Points of Intersection that do coincide. Which being premis'd, there arises several Cases of the 1st, 2d, 5th and 10th Corollaries, the Principal of which are as follow.

1. If there be two Tangents, KS, LS, meeting one another in the Point S, and two other right Lines MN, AR, parallel to those Tangents terminating in the Section, and meeting one another in P; I say, $MP \times PN : AP \times PR :: \overline{KS}^2 : \overline{LS}^2$. This has been prov'd in the Theorem, with regard to the Parabola: But for the other Sections, if in the first Corollary you conceive the Line FG to fall in the Tangent KS, and BD in LS; then it is plain, that the two Points of Intersection F, G, will coincide in the Point of Contact K; as likewise the two Points B, D, in the Point of Contact L; and so the Rectangles $FQ \times QG$, $BQ \times QD$, will become the Squares $\overline{KS}^2$, $\overline{LS}^2$.

2. In an Ellipsis, or the opposite Sections, if there be drawn a Tangent TX parallel to KS, and meeting SL in the Point X; then we prove, as in *Num.* 1. that $MP \times PN : AP \times PR :: \overline{TX}^2 : \overline{LX}^2$; and therefore $\overline{KS}^2 : \overline{LS}^2 :: \overline{TX}^2 : \overline{LX}^2$; and $KS : SL :: TX : LX$. That is, if two Tangents parallel to KS, TX, meet a third Tangent LS, in the Points S, X; then we shall have this Proportion, $KS : LS :: TX : LX$; or $KS : TX :: LS : LX$.

3. In the Ellipsis, Hyperbola, or opposite Sections, if there are two Tangents KS, LS, meeting one another in the Point S; and if the two Semidiameters CT, CZ, be drawn parallel to those Tangents; I say, these Tangents will be to one another, as the two Semidiameters. For by the Theorem, $\overline{CT}^2 : \overline{CZ}^2 :: MP \times PN : AP \times PR :: \overline{KS}^2 : \overline{LS}^2$ by *Num.* 1. therefore $CT : CZ :: KS : LS$.

4. If there be two right Lines AR, FG, terminating in a Conick Section, and meeting the two Tangents KI, LO, being parallel to them in the Points I, O; then I say, $FO \times OG : \overline{LO}^2 :: \overline{KI}^2 : AI \times IR$. This is evident by conceiving in *Corol.* 1. that the Line BD becomes the Tangent LO; and MN, the Tangent KI.

5. If there are two Parallels AR, BD, terminating in a Conick Section, and meeting the Tangent KH in the Points I, H; then I say, $\overline{KI}^2 : AI \times IR :: \overline{KH}^2 : BH \times HD$, or $\overline{KI}^2 : \overline{KH}^2 :: AI \times IR : BH \times HD$. This is a Continuation of *Corol.* 2. by conceiving the Line FG to fall in the Tangent KH.

6. If the Conick Section, in the last Number, be suppos'd an Hyperbola, and the Tangent HK an Asymptote; then the Rectangles $BH \times HD$, $AP \times IR$, will become equal to one another. For then the

* Art. 108. the Point of Contact K will be * infinitely distant from the Points H, I; and consequently the infinite right Lines HK, IK, whose Difference is only the finite Line HI, may be taken as Equals. This has been demonstrated already in the 97th Article; and the Reason why we have repeated it again, is only for proving what we have laid down, and shewing that the same Truths may be discover'd after very different ways.

7. If there be two Tangents KS, LS, meeting one another in the Point S; and if there be a right Line AR, terminated by the Section, parallel to one of those Tangents, as LS, and meeting the other KS, in the Point I; then I say, $\overline{KI}^2 : AI \times IR :: \overline{KS}^2 : \overline{LS}^2$. This is manifest by conceiving, in the second Corollary, the Lines FG, BD, to fall in the Tangents KS, LS.

8. In an Ellipsis, or the opposite Sections, if there be two parallel Tangents KI, TV, meeting the Line AR, terminating in the Section, in the Points I, V; then I say, $\overline{KI}^2 : AI \times IR :: \overline{TV}^2 : RV \times VA$. This is a Continuation of the second Corollary, by supposing the Parallels MN, FG, to fall in the Tangents TV, KI.

FIG. 94. 9. In a Parabola, if there are two Parallels MN, CH, one of which touches the Section in C, and the other is terminated by the same; and if through any two Points A, B, of the Section, there be drawn the two Diameters AF, BO, meeting the Lines MN, CH, in the Points F, O; then it is evident, by conceiving (in *Num.* 1, and 2. of *Corol.* 6.) EL to fall in the Tangent CH; 1. That $MF \times FN : \overline{CO}^2 :: AF : BO$. 2. That if FA be produced till it meets the Tangent CH in Q, then we shall have $MF \times FN : \overline{CQ}^2 :: AF : AQ$.

FIG. 98. 10. If there be two Parallels MN, KT, whereof one, as KT, touches an Hyperbola in K, and meets one of the Asymptotes in S, and the other MN is terminated by either of the opposite Sections, and meets the same Asymptote in Q; and if, thro' any two Points A, B, in the Section, there be drawn two Parallels AP, BT, to the Asymptote CS, meeting the Lines MN, KT, in the Points P, T; then we shall have (by supposing in the three Numbers of *Coroll.* 10. the Secant GH to fall in the Tangent RT) 1. The Rectangle $MP \times PN : \overline{KT}^2 :: AP \times PQ : BT \times TS$. 2. By producing PA till it meets KT in R, the Rectangle $MP \times PN : \overline{KR}^2 :: AP : AR$. 3. The Square $\overline{KT}^2 : \overline{KR}^2 :: BT \times TS : AR \times RS$.

11. In the opposite Sections, if there be two parallel Tangents KR, LF, meeting the Asymptote CS in the Points S, V; and if thro' any two Points A, B, in the Section, there be drawn two Parallels AR, BF to the Asymptote CS, meeting the two Tangents in the Points R, F; then we shall have (by supposing, in *Numb.* 1, and 2. *Corol.*

Corol. 10. the two Secants MN, GH, to fall in the two Tangents KR, LF) 1. The Square $\overline{KR}^2 : \overline{LF}^2 :: AR \times RS : BF \times FV$. 2. The Square $\overline{KR}^2 : \overline{LE}^2 :: AR : AE$.

PROPOSITION XIV.

Problem.

176. *TO describe an Ellipsis, or two opposite Hyperbola's about a given Parallelogram* FGHK; *so that one of the Diameters* AB *thereof, being parallel to the two Sides* FK, GH, *be to the Conjugate Diameter* DE, *in the given Ratio of* m *to* n. FIG. 99, 100, *and* 101.

Draw the Lines AB, DE, bisecting the opposite Sides of the given Parallelogram $FGHK$; then it is * manifest, that those Lines will be the two Conjugate Diameters of the Section requir'd, and their Point of Intersection will be the Centre thereof; because by one of the Conditions of the Problem, the two Parallels FG, KH, must terminate in the Section, as well as the two other Parallels FK, GH. This being laid down, if AB, DE be taken for those two Conjugate Diameters; and you call the given Quantities CL, or CO, a; LF, or OK, b; and the unknown Quantity CA or CB, t; then we shall have, 1. (when the Section is * an Ellipsis) $BL \times LA\ (tt - aa) : \overline{LF}^2\ (bb) :: AB : DE :: mm : nn$. And therefore $tt = aa + \frac{mmbb}{nn}$. *Art. 146. *Art. 41, and 55.

2. When * the Sections are opposite ones, $\overline{CL}^2 \mp \overline{CA}^2\ (aa \mp tt) : \overline{LF}^2\ (bb) :: \overline{AB}^2\ \overline{DE}^2 :: mm : nn$: and so tt is $= aa - \frac{mmbb}{nn}$, or $tt = \frac{mmbb}{nn} - aa$; viz. $= aa - \frac{mmbb}{nn}$, when the Line AB is a first Diameter, and $= \frac{mmbb}{nn} - aa$, when the same is a second Diameter; and from hence arises the following Construction, of which I make three Cases. * *Art.* 81, *and* 118.

Case 1. When the Section is an Ellipsis, make the right-angled Triangle FST such; that the Side ST be $= CL$, and the Side $SV = \frac{m}{n} LF$; then if an Ellipsis be describ'd with the Semidiameter $CA = TV$, having the same Proportion to its Semi-conjugate CD, as m is to n. I say, that Ellipsis will answer the Problem. For 1. The Diameter AB being parallel to the Sides FK, GH, is to the Conjugate DE, in the given Ratio of m to n. 2. Because the Triangle TSV, is right-angled at S, the Square $\overline{TV}^2$ or $\overline{CA}^2\ (tt) = \overline{TS}^2\ (aa) + \overline{SV}^2$

mmbb

$\left(\frac{mmbb}{nn}\right)$; and therefore $BL \times LA\ (tt-aa) = \frac{mmbb}{nn}$; and consequently we have $BL \times LA \left(\frac{mmbb}{nn}\right) : \overline{LF}^2\ (bb) :: mm : nn :: \overline{AB}^2 : \overline{DE}^2$. Whence it appears, that LF is an Ordinate to the Diameter AB; and so the Section passes thro' the Point F. After the same manner we demonstrate that the Section will pass thro' the three Points G, H, K; because $GL = LF = OK = OH$, and $CO = CL$.

Case 2. When the Sections are opposite ones, and CL is greater than $\frac{m}{n}$ LF: form the Right-angl'd Triangle TSV, such, that the Side SV be $= \frac{m}{n} LF$, and the Hypothenuse $VT = CL$; and then describe two opposite Hyperbola's, having the Semi-first Diameter $CA = TS$, in the same Proportion to CD, the Semi-conjugate, as m is to n

Case 3. When the Sections are opposite ones, and CL is less than $\frac{m}{n} LF$: Then the Right-angl'd Triangle TSV must be form'd, having the Side $TS = CL$, and the Hypothenuse $SV = \frac{m}{n} LF$. And afterwards two opposite Hpperbola's must be described with the Semi-second Diameter $CA = TV$, and the Semi-conjugate CD, having the same proportion to one another as m to n.

The Demonstration of these two last Cases is the same as that of the first; but it must be observed when CL is $= \frac{m}{n} LF$, the Problem is impossible.

COROLLARY. I.

177. BEcause the position of the two Conjugate Diameters AB, DE, is determinate, as well as their Magnitude; since by the Conditions of the Problem, the said Diameters must cut the opposite Sides of the Parellelogram, and there can be found but one Value for the Semi-diameters CA, CB; therefore the Problem can have but one Answer.

COROLLARY II.

178. HENCE appears the Manner of describing a Conick Section, about a given Parallelogram $FGHK$, and passing through a given Point M.

For draw two Conjugate Diameters AB, DE, bisecting the opposite Sides of the Parallelogram; also from the given Point M draw the

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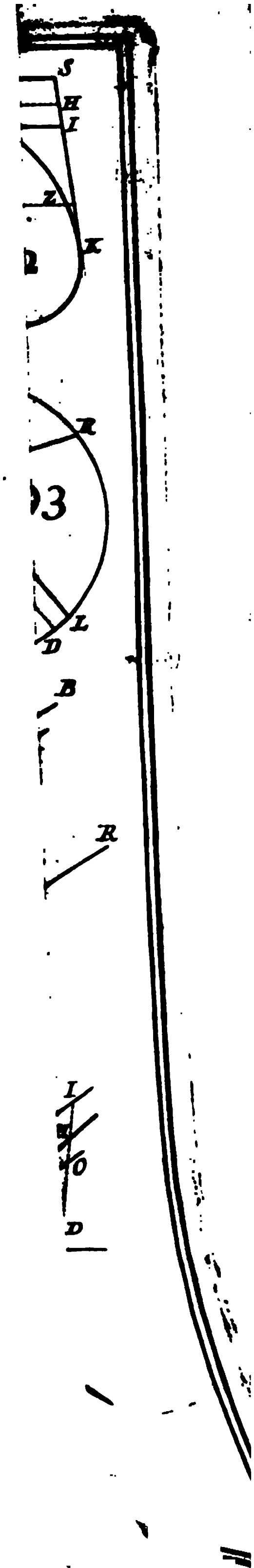
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D

$\left(\frac{mmbb}{nn}\right)$; and therefore $BL \times LA\,(tt-aa)=\frac{mmbb}{nn}$; and consequently we have $BL \times LA\left(\frac{mmbb}{nn}\right) : \overline{LF}^2\,(bb) :: mm : nn :: \overline{AB}^2 : \overline{DE}^2$. Whence it appears, that LF is an Ordinate to the Diameter AB; and so the Section passes thro' the Point F. After the same manner we demonstrate that the Section will pass thro' the three Points G, H, K; because $GL=LF=OK=OH$, and $CO=CL$.

Case 2. When the Sections are opposite ones, and CL is greater than $\frac{m}{n}LF$: form the Right-angl'd Triangle TSV, such, that the Side SV be $=\frac{m}{n}LF$, and the Hypothenuse $VT=CL$; and then describe two opposite Hyperbola's, having the Semi-first Diameter $CA=TS$, in the same Proportion to CD, the Semi-conjugate, as m is to n.

Case 3. When the Sections are opposite ones, and CL is less than $\frac{m}{n}LF$: Then the Right-angl'd Triangle TSV must be form'd, having the Side $TS=CL$, and the Hypothenuse $SV=\frac{m}{n}LF$. And afterwards two opposite Hpperbola's must be described with the Semi-second Diameter $CA=TV$, and the Semi-conjugate CD, having the same proportion to one another as m to n.

The Demonstration of these two last Cases is the same as that of the first; but it must be observed when CL is $=\frac{m}{n}LF$, the Problem is impossible.

COROLLARY. I.

177. BEcause the position of the two Conjugate Diameters AB, DE, is determinate, as well as their Magnitude; since by the Conditions of the Problem, the said Diameters must cut the opposite Sides of the Parellelogram, and there can be found but one Value for the Semi-diameters CA, CB; therefore the Problem can have but one Answer.

COROLLARY II.

178. HENCE appears the Manner of describing a Conick Section, about a given Parallelogram $FGHK$, and passing through a given Point M.

For draw two Conjugate Diameters AB, DE, bisecting the opposite Sides of the Parallelogram; also from the given Point M draw the

Ordinate MP to the Diameter AB, meeting the opposite Sides FK, GH, in the Points R, Q, and the Section (which I suppose to be describ'd) in the Point N; then it is evident that $PN = PM$, and so $RN = QM$, since $PR = PQ$. Therefore the Rectangle $RM \times MQ$ will be equal to the Rectangle $RM \times RN$. But * $FR \times RK : MR \times RN :: RM \times MQ :: \overline{AB}^2 : \overline{DE}^2$. And consequently the Ratio of the Diameter AB, (which is parallel to the Sides FK, GH,) to the Conjugate DE is given; because the Rectangles $FR \times RK$, $RM \times MQ$, are given. Farther, the Section will be an Ellipsis, when that of the two Ordinates (MP, KO) to the Diameter (AB) falling on the same Side the Centre C, which is nighest the Centre, is greater than that more distant; and contrariwise, the opposite Sections, when that Ordinate is lesser. Whence it appears, that this Question is brought to that of the last Problem. *Art. 164.

If the given Point M should fall in one of the Sides of the given Parallelogram, continued out at Pleasure; then the Problem would be impossible; because that Side would meet the Section in three Points, which * cannot be. *Art. 149.

COROLLARY. III.

179. HENCE arises a Method of describing a Conick Section such, that the right Line AB, given in Position, may be one of its Diameters, the given Point C the Centre thereof, and the right Lines MP, KO, Ordinates to that Diameter.

In the Diameter AB, assume CL, equal to CO, and draw LF parallel and equal to OK; then it is evident, * that LF will be an Ordinate to the Diameter AB, and so producing KO to H, and FL to G, so that OH be $= OK$, and $LG = LF$; the equal and parallel right Lines KH, FG, will be * double Ordinates to the Diameter AB. Whence it appears, that the Section must be describ'd about the Parallelogram $FGHK$, and must pass through the given Point M. And this may be done by the last Corollary. *Art. 49, 55, 85, and 118. *Art. 144.

Because this Question is brought to that of the last Corollary, which is brought to the Problem in *Coroll.* 1. and since this Problem has but one Answer, therefore there can be but one Answer that will fulfil the Conditions of this Corollary.

PROPOSITION XV.

Problem.

180. *TO describe a Conick Section, which shall pass thro' five given Points* F, M, K, G, N; *and to demonstrate that there can be but one Section that will answer the Problem.* FIG. 102, 103.

O Join

Join four of the given Points by the two right Lines FG, MN, meeting one another in the Point R; and through the fifth given Point K, draw two right Lines KD, KH, parallel to FG, MN, meeting them in the Points E, Q; then in those Parallels produc'd, (if necessary) assume the Points D, H such, that $MR \times RN : GR \times RF :: ME \times EN : KE \times ED$; and $FR \times RG : MR \times RN :: FQ \times QG : HQ \times QK$. (Observing that the Points K, D, or K, H, must fall on each Side the Point of Concurrence E or Q, when the Points M, N, or F, G, likewise fall on each Side that said Point, and contrariwise.) This being done, draw the right Lines LI, AB, through the middle Points of the Parallels DK, FG, and MN, KH, meeting one another in the Point C; then if a Conick Section be de-

* Art. 179. scrib'd *, with the Line AB given in Position as a Diameter, the given Point C as the Centre, and the two right Lines MP, KO, as Ordinates to that Diameter: I say, that Section only will satisfy the Conditions of the Problem.

* Art. 165. For the two Points D, H, will be * in the Section passing through the five given Points F, M, K, G, N; and so the Lines LI, AB, will

* Art. 146, and 147. be * Diameters, whose Intersection shall determine the Centre C. Therefore it is manifest, that the Line AB given in Position, will be a Diameter of the Conick Section passing through the five given Points, the Point C the Centre, and the Lines MP, KO, Ordinates to the Diameter AB. And because there is but one Section only that can fulfil these Conditions, it is manifest, that the same will be that requir'd, and there is no other but that.

If it happens that the Diameters AB, LI, be parallel to one another; then the Section will be * a Parabola, which may be describ'd

* Art. 147. by Art. 170.

The End of the Fourth Book.

BOOK

BOOK V.

Of the Comparison of the Conick Sections, and their Segments, with each other.

LEMMA I.

181. *IF the Difference of two Quantities does continually diminish, so that at last it becomes less than any given Quantity; then will those two Quantities at last be equal.*

For if the two Quantities at last be not equal, we may assign some Difference between them; which is contrary to the Hypothesis.

LEMMA II.

182. *IF the Ratio of two Quantities be such, that the Antecedent remaining always the same, while their Difference and the Consequent diminishing continually, do become less than any given Quantity; then will those two Quantities at last be equal.*

For the Antecedent will be equal to its Consequent by the last Lemma; and so the Quantities, whose Ratio they express, will be equal.

LEMMA III.

183. *IF an Arc* (MN) *of any Curve Line* ABG *be suppos'd to be infinitely small, that is, less than any given Quantity; and if thro' the Extremities of that Arc there be suppos'd to be drawn the Ordinates* MP, NQ, *to the Axis or Diameter* AC, *together with the Parallels* MR, NS *to that Diameter: I say, each of the Parallelograms* PQRM, PQNS, *may be taken for the Space* PQNM, *contained between the Ordinates* PM, QN, *the very small right Line* PQ, *and the very small Curve* MN. FIG. 104.

All the Points of any Curve do either continually recede from its Diameter, or continually approach thereto; or else that Curve Line is composed of several Parts; some of which recede more and more from, and others approach nearer and nearer to the Diameter.

For

For it is evident, that there is no Part of a Curve Line that can have all its Points equally diftant from the Diameter; becaufe that Part then would not be curv'd, but a ftraight Line parallel to that Diameter.

Let us fuppofe, 1. That the Arc MN be taken in the Curve AMB, all of whofe Points recede more and more from the Diameter AC; then if the Arc MO be taken of a finite Magnitude from M beyond the Point N, and the Ordinate OF be drawn parallel to MP, as likewife the right Lines OD, ME, parallel to the Diameter AC; it is manifeft, that the Curve-lin'd Space $PFOM$, will be greater than the infcrib'd Parallelogram $PFEM$, and lefs than the circumfcrib'd Parallelogram $PFOD$. Now if the Point O be fuppos'd to move along the Curve towards M, it is plain, that the Parallelogram $MEOD$, which is the Difference of the circumfcrib'd and infcrib'd Parallelograms to the Arc OM, will diminifh continually, till at laft it will become nothing, at the Inftant the Point O comes to M: Therefore, when the Point O is come to N, that is, infinitely near to M, then the Parallelogram $MEOD$, which becomes $MRNS$, will be lefs than any given Magnitude; and fo, by *Lemma* 1. the Parallelograms $PQRM$, $PQNS$, will then become equal to one another; and confequently each of them equal likewife to the curvilineal Space $PQNM$. Therefore, &c.

2. Suppofe the very fmall Arc MN, to be taken in the Curve BMG, all the Points of which approach nearer and nearer to the Points of the Diameter CG; then it is manifeft, that the Demonftration in this Cafe will be the fame as in the laft, only here the circumfcrib'd Parallelogram $PQNS$, becomes an infcrib'd one.

3. Let the Curve ABG be compos'd of feveral Portions, fome of which, as AB, recede more and more from the Diameter AG; and contrariwife others, as BG, accede nearer and nearer to it; then, I fay, the Points, as B, that feparate thofe Parts, cannot fall in the Arcs MN; for if this were poffible, the Point B would be nearer to M than the Point N is; which is contrary to the Suppofition; and fo it is manifeft, that this laft Cafe is contain'd in either the firft or fecond Cafe.

COROLLARY I.

184. HENCE, if any Ordinate CB be drawn at pleafure parallel to PM; and if the Part of the Curve AB, be fuppos'd to be divided into an infinite Number of infinitely fmall Arcs, as MN; then the Space ACB, contain'd under the right Lines AC, CB, and the Part of the Curve AB, will be equal to the Sum of all the Parallelograms, as $PQRM$, or $PQNS$. It is evident farther, that the Space

Space $MPCB$, contained under the Right Lines MP, PC, CB, and the Part of the Curve MB will be equal to the Sum of all the Parallelograms ($PQRN$) that can be form'd in that Space; and the same is to be understood of the whole Curve ABG.

Corollary II.

185. If there be any Figure $CMDOC$, included between two Parallels CE, DF; and if any where between those Parallels there be suppos'd two right Lines MO, NL, infinitely near to one another, and likewise parallel to CE, DF; then, I say, the Space or Part ($OMNL$) of the Figure $CMDOC$ included between those Parallels, will be equal to the Rectangle under one of them, as MO, and their Distance MR, or OS. For if the Perpendicular AB be drawn to the Parallels CE, DF, meeting the Parallels MO, NL, in the Points P, Q; then it is manifest, by the Lemma *, that the Space $PMNQ$ is equal to the Rectangle $PMRQ$, and the Space $POLQ$ to the Rectangle $POSQ$; and consequently the Space $OMNL$ is equal to the Rectangle $OMRS$, or $OM \times PQ$. Fig. 105. * Art. 183.

Corollary. III.

186. It is manifest by the last Corollary, that if there be any two Figures $CMDOC$, $EGFHE$, included between two Parallels CE, DF, being such, that if a right Line MH be any where drawn between CE, DF, parallel to them; the Parts MO, GH, of that Line included by in Figures $CMDOC$, $EGFHE$, be in a given Ratio; then, I say, the two Figures, that is, the Spaces contain'd between them are also to one another in a given Ratio. For supposing another Parallel NK infinitely near to MH, and drawing a Perpendicular AB to the Parallels CE, DF, meeting the Parallels MH, NK, in the Points P, Q; then it is evident, (by the last Corollary) that the Space $OMNL$ is equal to the Rectangle $OM \times PQ$. And moreover, the Space $GHKI$ equal to the Rectangle $GH \times PQ$. Therefore those two Spaces will be to one another, as MO to GH; and because this happens always, let the Line MH be drawn in any Place: It follows, that the Sum of all the small Spaces $MNLO$, that is, the Space $CMDOC$ will be to the Sum of all the small Spaces $GHIK$, that is, to the Space $EGFHE$, in a given Ratio.

After the same manner we prove, that the Part (MDO) of the Figure $CMDOC$, is to the correspondent Part (GFH) of the other Figure $EGFHE$, in a given Ratio; as also the remaining Parts CMO, EGH.

It

Hence, if the given Ratio be a Ratio of Equality, that is, if the Parts (MO, GH) of the right Line MH, are always equal to one another; then the Spaces $CMDOC$, $EGFHE$, and their correspondent Parts MDO, GFH, and CMO, EGH, will be equal to one another.

Lemma IV.

Fig. 106. 187. *If* MN *be an infinitely small Arc of any Curve Line; and if* MT, NT, *be two Tangents meeting one another in the Point* T, *and if the Line* MN *be a Chord, and the right Line* NS *be drawn perpendicular to the Tangent* MT *produced; then, I say, the Arc* MN *may be taken for its Chord* MN, *or for the Sum of the two Tangents* MT, NT; *or else for the right Line* MS.

Every Cure is concave necessarily one way, or made up of many Parts, whereof some are concave one way, and the others concave the contrary way. Now the Points separating those Parts cannot be

*Art. 183. * had in infinitely small Arcs, as MN; because then they would be nearer to the Point M than N is; which is contrary to the Supposition. Therefore we can suppose always, that the Arc MN is a Part of a Curve, that is concave one way.

Now if the Arc MO be taken in the Curve having a finite Magnitude; and if there be drawn the Chord OM, the Tangent OG, and the right Line OD parallel to NS; then it is manifest, 1. That the Tangent MD is less than the Chord MO, because the Triangle MDO is right-angled at D, and consequently the same is lesser still than the Arc MNO; wherefore the Arc MNO, and its Chord MO, are each greater than MD, and each of them less than the Sum of the two Tangents MG, OG. 2. Because the Arc MNO is concave but one way, therefore if a Tangent TR be drawn through any Point N in the Arc MO; the Points T, R, wherein that Tangent meets the Tangents MG, OG, will be situate between the Points M, G; and O, G; and so the Angle OGD, is greater than the Angle RTG, or NTS.

This being suppos'd, if the right Lines ME, MF, be drawn parallel to OG, NT, meeting the right Line DO in the Points E, F; and if the Point O be suppos'd to move along the Curve towards the Point M; then it is visible, that the Angle OGD, or the Angle EMD, will diminish continually, until it vanishes at the same time as the Point O comes to M; because then the Tangent OG will coincide with the Tangent MD; and therefore the Line ME diminishes continually, until it becomes at last equal to MD. Therefore, when the Point O is come to N, that is, when the same is infinitely near to the Point M; the Line ME, being then in MF, will differ from the Tangent MD, only by a

2 Magni-

Magnitude less than any given one. And so the Lines TN, TS, * will be equal to one another. Therefore the two Tangents MT, TN, taken together, will be then equal to the Right Line MS, or to the Arc MN, or else to the Chord MN. *W. W. D.* *Art. 182.

COROLLARY I.

188. BEcause the Angle FMD, or NTS, equal to it, is infinitely small, in the Supposition of the Point N being infinitely near to the Point M; therefore the Internal Angle NMT, of the Triangle MTN, being less than the External Angle NTS, will be also infinitely small, that is, less than any given Angle: And so there can no Right Line be drawn through the Point M, in the Angle TMN. Whence it appears that the two Right Lines MT, MN, do coincide; and so any Tangent may be look'd upon as a Right Line passing thro' two Points of a Curve Line infinitely near to each other.

COROLLARY II.

189. IF any Curve Line be supposed to be divided into an infinite Number of infinitely small Arcs as MN: Then it is evident, that if the Chords of those Arcs be taken for the Arcs themselves, there will arise a Polygon of an infinite Number of Sides, each Side being infinitely small, which may be taken for that Curve Line, as no wise differing there-from. * And farther, if the small Sides of that Polygon be produced both ways, they will be Tangents to that Curve; since every of them do pass through two Points infinitely near to each other. *Art. 187.

SCHOLIUM.

190. IT must here be observ'd, that what I have said with regard to the Tangents to the Conick Sections, does not extend to any other Curves but those that are Concave the same way, as are the Conick Sections: * Whereas that Definition of Tangents laid down above, takes in the Tangents to all Curves in general, and is the Foundation of the Method for drawing Tangents, explain'd in my Treatise, *Des Infiniment petits*, which I dare affirm is as simple and general as can be desired. A short Specimen of which may be seen at the End of this Book. * Art. 26 61, 124.

DEFI-

DEFINITIONS.

1.

FIG. 107, 108, 109. Two Segments of any two Curve Lines *B A d*, *b a d*, are called *similar*, if when any Right-lin'd Figure *B M N O D*, being inscribed within one of them, we can inscribe always a similar Right-lin'd Figure *b m n o d*, in the other.

2.

Two Conick Sections are said to be *similar*, when any Segment *B A D*, being taken in the one, we can assign always a similar Segment *b a d* in the other.

3.

The Diameters *A P*, *a p*, in two Conick Sections, are said to be *similar*, when they make the same Angles *A P M*, *a p m*, with their Ordinates *P M*, *p m*.

COROLLARY.

191. HENCE it appears that the lesser the Sides *B M*, *M N*, &c. *b m*, *m n*, &c. are, the greater is their Number, and the nearer do the similar Right-lin'd Figures *B M N O D*, *b m n o d*, approach to the Segments *B A D*, *b a d*, in which they are inscrib'd; so that at last the said similar Right-lin'd Figures will become equal to those Segments, *viz.* when each * of the Sides is infinitely small, and their Number consequently infinite. Therefore the similar Segments *B A D*, *b a d*, are to one another as the Squares of their Chords, *B D*, *b d*, being Homologous Sides; and the Parts of the Curves *B A D*, *b a d*, as these Chords.

* *Art.* 189.

PROPOSITION I.

Theorem.

FIG. 107. 192. *IF there be two Parabolas* A M, a m, *having two similar Diameters* A L, a L, *situate in the same straight Line: So that the Ordinates* P M, p m, *be parallel to one another*; *and if the fix'd Point* L *be assum'd in that Line within the Parabola, so that* L A *be to* L a, *as the Parameter* (A G) *of the Diameter* (A L) *of the Parabola* A M, *to the Parameter* (a g) *of the Diameter* (a L) *of the Parabola* a m: *Then I say, if a Right Line* L M, *be drawn from the fix'd Point* L *to any Point* M *of the Parabola* A M; *that Line will meet the other Parabola* a m, *in one Point* m, *being such that* L M : L m : : L A : L a.

I Draw

Draw the Ordinate MP, and call the given Quantities LA, a; La, b; AG, p; and the indeterminate Quantities AP, x; PM, y: then we shall have this Proportion, *viz.* $LA\ (a) : La\ (b) :: AG\ (p) : ag = \frac{bp}{a}$. But if ap be taken in the Diameter (aL) of the Parabola am equal to $\frac{bx}{a}$, and the Ordinate pm be drawn; then it is plain, * that $\overline{pm}^2 = pa \times ag \left(\frac{bbpx}{aa}\right) = \frac{bbyy}{aa}$ by putting yy for px *; and so pm is $= \frac{by}{a}$. Therefore $PM\ (y) : pm \left(\frac{by}{a}\right) :: LP\ (a-x) : Lp\ (b - \frac{bx}{a})$. And consequently the Line LM will pass through the Point m, the Extremity of the Ordinate pm, that is, the said Line will cut the Parabola am in that Point: And so because the Triangles LPM, Lpm, are similar, therefore $LM : Lm :: PM\ (y) : pm \left(\frac{by}{a}\right) :: LA\ (a) : La\ (b)$. *W. W. D.*

* Art. 6, and 20.
* Ibid.

Corollary. I.

193. If any Segment BAD, be taken in the Parabola AM; and if the right Lines LB, LD, be drawn, meeting the other Parabola am in the Points b, d, and the Chord bd be drawn; then, I say, the Segment (bad) of the Parabola am, is similar to the Segment BAD of the Parabola AM. For if any right-lin'd Figure $BMNOD$ be inscrib'd in the Segment BAD; then, by drawing the right Lines LM, LN, LO, meeting the other Parabola in the Points m, n, o; it is manifest, that the Triangles LBM, Lbm; LMN, Lmn; LNO, Lno; LOD, Lod; LBD, Lbd, will be similar: and so the Sides BM, bm; MN, mn; NO, no; OD, od; BD, bd, will be parallel, and always in the same Ratio, each to its Correspondent; because all the right Lines LB, LM, LN, LO, LD, are divided in the same Ratio in the Points b, m, n, o, d: therefore the right-lin'd Figures $BMNOD$, $bmnod$, are similar. And since it is manifest, that this Demonstration is the same, let the right-lin'd Figure inscrib'd in the Segment BAD be what it will; therefore the Segments BAD, bad, are * similar; and so consequently are * the Parabola's Am, am, also.

* Def. 1.
* Def. 2.

COROLLARY II.

194. HENCE, if a double Ordinate EF be drawn in the Parabola AM, meeting the other Parabola am in the Points e, f; then the Segments (EAF, eaf,) of the Parabola's AM, am, will be similar.

COROLLARY III.

195. ALL Parabola's are similar; for if AL, aL be taken in the Diameters of two different Parabola's, being to one another as the Parameters AG, ag; and if the Diameter La be suppos'd to be plac'd in the Diameter LA, so that the Points L, L, coincide with one another, and the Ordinates PM, pm, be parallel; then, by drawing a right Line LM, from the fixed Point L to any Point M of the Parabola AM, it is manifest, that this Line will meet the other Parabola am in one Point m, being such, that $LM : Lm :: LA : La$.
*Art. 193. Therefore, * &c.

COROLLARY IV.

196. HENCE, if AL, aL, be taken in the Diameters of two different Parabola's, being to one another as the Parameters of those Diameters; and if the double Ordinates EF, ef, be drawn through the Points L, L; then the Segments (EAF, eaf,) of the two Parabola's AM, am, will be similar.

COROLLARY V.

197. IF two Segments BAD, bad, are similar, and one of them, as BAD, be the Segment of some one Parabola; then, I say, the other Segment bad will be a Segment of some other Parabola; and so among all the Curves possible, there can be none but Parabola's that can be similar to a given Parabola. For if the small Segment bad be so placed within the great one BAD, that the Chords bd, BD, be parallel; and if any two similar right-lin'd Figures $BMNOD$, $bmnod$, be inscrib'd in those Segments; then it is plain, that the homologous Sides BM, bm; MN, mn, &c. of the said two Figures will be parallel, because the Angles DBM, dbm; BMN, bmn, &c. are equal. And drawing LM, LN, LO, thro' L the Point of Concurrence of the right Lines Bb, Dd, which join the Ends of the parallel Chords BD, bd, (being the two homologous Sides given); then the said Lines LM, LN, LO, will pass through the

the correspondent Points m, n, o, wherein they will be divided in the same Ratio as LB is in b, or LD in d; because $BD : bd :: LB : Lb :: BM : bm :: LM : Lm :: MN : Mn :: LN : Ln :: NO : no :: LO : Lo :: OD : od$.

Now if the Diameter (LA) of the Parabola AM be drawn thro' the Point L; and if the same Parabola am be divided in a, in the same Ratio as LB is in mb, or LD in d; and if you describe * the Parabola am with the Diameter aL, and the Parameter ag, (being to the Parameter AG of the Diameter (AL) of the Parabola AM, as La is to LA,) whose Ordinates pm are parallel to the Ordinates PM of the other Parabola; then it is * manifest, that the said Parabola will pass through all the Points b, m, n, o, d, dividing all the right Lines LB, LM, LN, LO, LD, in the given Ratio of BD to bd. And because the same Reason holds, let the Number of Sides of the similar right-lin'd Figures $BMNOD$, $bmnod$, as likewise their Magnitude, be what they will; therefore the Parabola am passes thro' every Point, as the Segment bad does; and so that Segment will be a Part thereof. *W. W. D.*

*Art. 161. *Art. 192.

PROPOSITION II.

Theorem.

198. *If there be an Ellipsis, or Hyperbola* AM, *one of whose first Diameters is the Line* AH, *and the Line* AG *its Parameter; and if the fix'd Point* L *be taken in that Diameter (produced in the Hyperbola) as also the Points* a, h, *such that* LA : LH :: La : Lh. *And again, if there be another Ellipsis or Hyperbola, whose first Diameter is the Line* ah, *and Parameter to that Diameter the Line* ag, *which is to* AG, *as* ah *to* AH; *and if the Ordinates* pm *to the same be parallel to the Ordinates* PM *of the other Section* AM: *Then if any right Line* LM, *be drawn from the fixed Point* L *to any Point* M *in the Section* AM, *I say, that Line will meet the other Section* am *in the Point* m, *so that* LM : Lm :: LA : La : *that is, all right Lines drawn from the fixed Point* L *to the Section* AM, *will be divided in the same Ratio by the Section* am. FIG. 108, 109.

We are to prove, that LM : Lm :: LA : La.

Draw the Ordinate MP, and call the given Quantities LA, a; La, b; AH, $2t$; and the indeterminate Quantities AP, x; PM, y: then we shall have this Proportion, $LA\,(a) : La\,(b) :: LH : Lh :: LH \pm LA$, or $AH\,(2t) : Lb \pm La$, or $ab = \frac{2bt}{a}$. Now if ap be taken in the Diameter (ab) of the Section am equal to $\frac{bx}{a}$, and the Ordinate

* Art. 42, 55, 81, and 118. dinate pm be drawn; then it is * manifest, that $AP \times PH$ $(2ax \mp xx)$ $: \overline{PM}^2$ (yy) $:: AH : AG :: ab : ag :: ap \times pb$ $\left(\frac{2abbx \mp bbxx}{aa}\right)$ $: \overline{pm}^2 = \frac{bbyy}{aa}$, and so $pm = \frac{by}{a}$. Therefore PM (y) $: pm$ $\left(\frac{by}{a}\right)$ $:: LP$ $(a-x)$ $:$ Lp $\left(b - \frac{bx}{a}\right)$. And consequently the Line LM will pass through m the Extremity of the Ordinate pm, that is, the said Line will cut the Section am in that Point. Therefore, because the Triangles LPM, Lpm, are similar, $LM : Lm :: PM$ (y) $: pm$ $\left(\frac{by}{a}\right)$ $:: LA$ (a) $: La$ (b). *W.W.D.*

Corollary I.

199. IF BAD be any Segment of the Section AM, and the right Lines LB, LD, be drawn meeting the other Section am in the Points b, d; and if the Chord bd be drawn likewise; then, I say, the Segment (bad) of the Section am, is similar to the Segment (BAD) of the Section AM; and therefore, if a double Ordinate EF be drawn in the Section AM, thro' the Point L, meeting the other Section in the Points e, f; the Segments EAF, eaf, of two Ellipses or Hyperbola's AM, am, will be similar. This may be demonstrated after the same manner, as is done for the Parabola in Art. 193, and 194.

Corollary II.

200. ALL Ellipses or Hyperbola's AM, am, having two similar Diameters AH, ab, in the same Ratio with their Parameters AG, ag, are similar to one another. For if AL, aL, be taken in the same Ratio as the Diameters AH, ab; and if the Diameter ab be suppos'd to be laid in AH, so that the Points L, L coincide; and the Ordinates pm, PM be parallel to one another; then the right Line LM being drawn from the fix'd Point L to any Point in one Section AM, it is evident, that the said Line will always meet the other Section am in the Point m, so that $LM : Lm :: LA : La$.
* Art. 199. Therefore, * &c.

Corollary III.

201. HENCE, if there be two Ellipses, or Hyperbola's AM, am, whose two similar Diameters AH, ab, have the same Ratio as their Parameters AG, ag; and if AL, aL, be taken in the same Ratio

Ratio as the Diameters Ab, ab, and the double Ordinates EF, ef, be drawn thro' the Points L, L; then it is manifest, that the Segments EAF, eaf, of the two Sections AM, am, are similar.

COROLLARY IV.

202. IF two Segments BAD, bad, be similar; and one of them be the Segment of an Ellipsis or Hyperbola AM, any one of whose Diameters is the Line AH, and AG the Parameter thereof; then, I say, the other Segment bad will be that of some other Ellipsis or Hyperbola am, having the Line ab similar to AH, for one of the Diameters thereof, and the Ratio of this Diameter to its Parameter ag, the same as that of the Diameter AH to its Parameter AG. For if the Segment bad be so plac'd within the Segment BAD, that the Chord bd be parallel to the Chord BD, and the Lines Bb, Dd, do concur in the Point L of the Diameter AH, (which is possible always) and if there be inscrib'd in both of the said Segments any similar right-lin'd Figures; then we can prove, as in the Parabola (*Art.* 197.) that the right Lines LM, LN, LO, will pass through the correspondent Points m, n, o, and will be divided by these Points in the same Ratio as LB is in b, or LD, in d.

Now if LA, LH, be divided in the same Ratio in the Points a, b, as LB is in b; and if an Ellipsis, or Hyperbola am, be describ'd * with the Diameter ab, and Parameter ag, (being to the Parameter AG of the Diameter AH, as La to LA, or ab to AH) whose Ordinates pm, be parallel to the Ordinates (PM) of the other Ellipsis or Hyperbola AM; then it is * evident, that that Section will pass through all the Points b, m, n, o, d, which divide all the right Lines LB, LM, LN, LO, LD, in the given Ratio of bd to BD. And because this Reasoning is the same always, be the Number of Sides of the right-lin'd similar Figures $BMNOD$, $bmnod$, as likewise their Magnitude, what they will: Therefore the Ellipsis, or Hyperbola am, passes through all the Points that the Segment bd does; and so that Segment is a Part or Portion of the same.

* *Art.* 161.
* *Art.* 198.

COROLLARY V.

203. HENCE, if there be two similar Ellipses, or Hyperbola's AM, am; and if AH be any Diameter of the Section AM; then we can have always a Diameter (ab) of the Section am, which is in the same Ratio to its Parameter ag, as AH to its Parameter AG; and so the similar Diameters AH, ab, will be in the same Ratio as the Diameters, which are Conjugates to them: And because there can be but two Pair of Conjugate Diameters * in an Ellipsis or Hyper-

* *Art.* 66, *and* 128.

Hyperbola making the same Angles with each other; and since the said Diameters do only differ in Position, their Magnitude being the same; therefore in similar Ellipses or Hyperbola's, all Conjugate Diameters making the same Angles, will be to one another in the same Ratio; observing to take the two greater of the Conjugate Diameters for the Antecedents of the Ratio's, and the lesser for the Consequents.

PROPOSITION III.

Theorem.

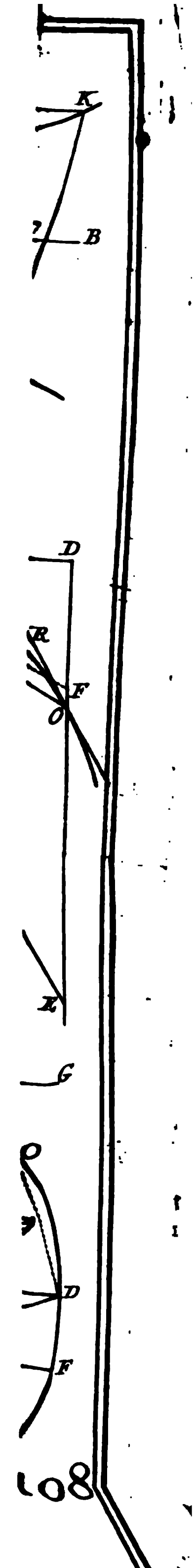
FIG. 110, 111. 204. *If any two Parallels* BD, EF, *be drawn in a Conick Section terminated by the same; and if their Extremities be joined by two right Lines* BE, DF; *I say, the Segments* BMEB, DMFD, *contain'd under Portions of the Section, and the right Lines joining the Extremities of the Parallels, will be equal to one another.*

Produce the Chords *BE, DF*, meeting each other in the Point *G*, and draw the right Line *GH* through the said Point *G*, and *(H)* the Middle of the Line *BD*; then the Line *GH* will bisect *EF* (parallel to *BD*) in the Point *K*; as also any other parallel (to the same Line *BD*) as *OO* in the Point *P*. Therefore the Line *HK*, will be
*Art. 146. a Diameter; * and the Parallels *BD, EF*, double Ordinates to the same; and so if a Line be drawn through any Point *P*, in that Diame-
*Art. 144. ter parallel to the Ordinates; that Line will meet * the Section in two Points *M, M*, equally distant from *P*. Whence it appears, that the Parts *MO, OM*, of any Parallel *(MM)* to *BD*, included within the Segments *BMEB, DMFD*, are always equal to one another, let that Parallel be any where drawn between the Lines *BD, EF*.
*Art. 186. Therefore it is manifest, * that those two Segments shall be equal.

If the Chords *BE, DF*, be parallel; then the right Line *HK* must be drawn thro' *H* the Middle of *BD*, parallel to those Chords, and the Demonstration will be the same as before.

COROLLARY I.

FIG. 110. 205. Because *PM* is equal always to *PM*, it follows, 1. That the Conick Trapezia *KHBE, KHDF*, are equal to one another. 2. (When the Line *BD* does not cut the Section, but touches it in the Point *A*) the Conick Trilineal Figures *AKE, AKF*, are equal; and so are likewise the Segments *AEMA, AFMA*: Because the Triangle *AEF*, is divided into two equal Parts by the Diameter *AK*, passing through the Middle of *EF*.


K
B
D
R
F
O
E
G
D
F
108

Corollary II.

. IF the Section be a Parabola, Ellipsis, or Hyperbola; and the right Lines BF, DE, be drawn through the Extremities of the allels BD, EF, cutting one another between those Parallels; n the Segments $BFDAB$, $DEBAD$, will be equal to one anor. For the Triangles BFD, BED, being between the same Pallels BD, EF, and having the same Base BD, are equal to one other; and so if the Segment $DMFD$ *plus* the Segment $BADB$ added to one Side, and the Segment $BMEB$ (which is equal to MFD) *plus* the same Segment $BADB$, to the other; then the holes $BFDAB$, $DEBAD$, will be equal to one another. Fig. 110.

Corollary III.

107. HENCE appears the manner of drawing two right Lines DG, DF, from a Point D in a Conick Section, such, that hey shall cut off from that Section two Segments $DGED$, $DFBD$, ach equal to a given Segment $BEDB$. For draw the right Lines BD, DE, and draw BG parallel to DE, and EF parallel to BD, neeting the Section in the Points G, F; then it is manifest, * by oining the right Line DF, that the Segment $DFBD$, is equal to he Segment $BEDB$; because DB, EF, are parallel. And in like nanner by joining DG, the Segment $DGED$ is equal to the Segment $BEDB$, because BG, DE, are parallel. Fig. 112. *Art. 106.

If the Point given happens to fall upon one End of the given Segment $DGED$; then a right Line GF must be drawn through the ther End, parallel to the Tangent passing through the Point D; which being done, if the Chord DF be drawn through the Point F, (wherein GF meets the Section,) and the given Point D; it is plain, hat the Segment $DFBD$ will be equal to the given Segment $DGED$.

Hence, in this last Case, there can be but one Segment $DFBD$, equal to the given Segment $DGED$; because any other Segment aving the given Point D, as one of the Extremes thereof, will be greater or less than the Segment $DFBD$, according as the other Extreme is nearer to, or farther from D than F is. Therefore, if two Segments $DGED$, $DFBD$, having one common Extreme D, are equal between themselves; and if a right Line be drawn through the Point D, parallel to the right Line GF, joining their other Ends; hen will that Line touch the Section in the Point D.

COROLLARY IV.

208. FROM the last Corollary arises a new and very easy way of drawing a Tangent through a Point D given in a Conick Section, which is as follows.

Draw any two right Lines DB, BE, through the Point D, meeting the Section in the Points B, E; moreover, through the Points B, E, draw the right Lines BG, EF, parallel to DE, BD, meeting the Section in the Points G, F, which join by the Line GF; then if a right Line be drawn thro' the Point D, parallel to GF, that Line will touch the Section in the Point D; because the Segments $DGED$, $DFBD$, being each equal to the Segment $BEDB$, will be equal to one another.

PROPOSITION IV.

Theorem.

FIG. 113, 114, 115. 209. *IN an Ellipsis, Hyperbola, or opposite Sections, if there be two right Lines* BD, EF *parallel between themselves, and terminating in the Section; and if the Semidiameters* CB, CE, CD, CF, *be drawn from the Centre* C: *I say, the Hyperbolick or Elliptick Sectors* CBE, CDF, *will be equal to one another.*

For if the Diameter CK be drawn through H, K, the Middles of BD, EF; then the Triangles CHB, CHD, and CKE, CKF, will be equal; because they have the same Vertex C, and their Bases HB, HD, and KE, KF, are equal. Therefore (*Fig.* 114.) $KHBE + CBE = CKE - CHB = CKF - CHD = KHDF + CDF$; and (*Fig.* 113, 115.) $KHBE - CBE = \pm CHB \mp CKE = \pm CHD \mp CKF = KHDF - CDF$. And because the Conick Trapezia
*Art. 205. $KHBE$, $KHDF$, are * equal; therefore the Elliptick or Hyperbolick Sectors CBE, CDF, are equal also.

COROLLARY I.

FIG. 113, 114. 210. IF the Section be an Ellipsis or Hyperbola; and if the Line BD, being parallel to EF, becomes a Tangent in A; then it is plain, that the Sectors CAE, CAF, will be equal to one another. For producing the Semidiameter CA, until it meets the Line EF in the Point K, the said Line EF will be bisected by the Point K, and consequently the Triangles CKE, CKF, will be equal. And
*Art. 205. the Trilineal Conick Figures AKE, AKF, are * so also. Whence, &c.

I

COROL-

COROLLARY II.

211. HENCE, if any Hyperbolick or Elliptick Sector CEF, be requir'd to be divided into two equal Parts; there is no more to do, but draw the Semidiameter CA, bisecting the Chord (EF) of that Sector in the Point K. From whence we can prove again, that the Sectors CBE, CDF, are equal to one another, if BD be parallel to EF. For since by this Means, the Sectors CAE, CAF, and CAB, CAD, are equal to one another: Therefore the Sectors CBE, CDF, being the Differences between them must needs be equal.

PROPOSITION V.

Theorem.

212. *If there be a Semicircle* ADH, *whose Diameter* AH *is the first Axis of an Ellipsis* ABH; *and if a Perpendicular be let fall from any Point* N *in the Periphery of the Circle to the Axis meeting the same in* P, *and the Ellipsis in the Point* M; *and lastly, if the right Lines* CM, CN, *be drawn from the Centre* C. *I say, the Elliptick Sector* CAM, *is to the circular Sector* CAN, *as* CB, *the half of the second Axis of the Ellipsis is to* CA *or* CD, *the half of the first Axis.* FIG. 116.

For $\overline{PM}^2 : \overline{CB}^2 :: AP \times PH : AC \times CH$, or $\overline{CA}^2$, by * the Property of the Ellipsis. And $\overline{PN}^2 : \overline{CD}^2 :: AP \times PH : AC \times CH$, or $\overline{CA}^2$, by the Property of the Circle; therefore $\overline{PM}^2 : \overline{CB}^2 :: \overline{PN}^2 : \overline{CD}^2$; or $\overline{PM}^2 : \overline{PN}^2 :: \overline{CB}^2 : \overline{CD}^2$. And extracting the square Roots, $PM : PN :: CB : CD$ or CA. And since this is so always, let the Perpendicular PMN fall any how; therefore * the whole Elliptick Space $ABHA$, is to the Semicircle $ADHA$, and the Part or Portion APM of that Space to the Part or Portion APN of the Semicircle, as CB to CD, or CA. But the right-angled Triangle CPM, is to the right-angled Triangle CPN, having the same Altitude, as the Base PM is to the Base PN, that is, as CB is to CD, or CA; and consequently the Elliptick Space APM *plus*, or *minus* the Triangle CPM (*viz. plus* when AP is less than AC, and *minus* when it is greater) that is, the Elliptick Sector CAM will be to the circular Space APN *plus* or *minus* the Triangle CPN, that is, to the circular Sector CAN, as CB is to CD, or CA. *W. W. D.*

* Art. 42, and 55.

* Art. 186.

Corollary I.

213. Because the Sector (CAN) of the Circle is equal to the Rectangle under the Arc AN, and one half the Radius CA, or CD: Therefore the Elliptick Sector CAM is equal to the Rectangle under the same Arc AN, and the one half of CB.

Corollary II.

214. If thro' any Point G, (besides P) in the first Axis (AH) be drawn a Perpendicular to that Axis, meeting the Ellipsis in the Point E, and the Circle in the Point F; I say, the Elliptick Sectors ACE, ACM, are to one another, as the circular Sectors ACF, ACN. For $ACM : ACN :: CB : CD$. And moreover $ACE : ACF :: CB : CD$. And therefore $ACM : ACN :: ACE : ACF$; and $ACM : ACE :: ACN : ACF$. Whence, if it be requir'd to find the Elliptick Sector ACM, which may be to the Elliptick Sector ACE, in a given Ratio, you need only find the circular Sector ACN, that may have that given Ratio to the Sector ACF, or else (which is the same thing) divide the Arc ANF, or Angle ACF, into that given Ratio.

PROPOSITION VI.

Theorem.

Fig. 117, 118. 215. *If there be two Hyperbola's* AM, AN, *or* BM, DN, *having the Point* C, *as a Centre common to them both, the right Line* CA, *a Semidiameter to them both, and any two right Lines* CB, CD, *situate in the same Line, semi-conjugate Diameters to* CA, CA; *and if through any Point* P *in the Semidiameter* CA *(produced, if necessary) there be drawn a right Line parallel to* CD, *meeting the Hyperbola's in the Points* M, N; *as also the right Lines* CM, CN, *from the Centre* C *to the said Points* M, N; *I say, the Hyperbolick Sectors* CAM, CAN, *or* CBM, CDN, *will be to one another, as the Semiconjugate Diameters* CB, CD.

* Art. 81, and 118. By the Property * of the two Hyperbola's AM, AN, or BM, DN, we have the two following Proportions, *viz.* $\overline{PM}^2 : \overline{CB}^2 :: \overline{CP}^2 \mp \overline{CA}^2 : \overline{CA}^2 :: \overline{PN}^2 : \overline{CD}^2$. And consequently $\overline{PM}^2 : \overline{PN}^2 :: \overline{CB}^2 : \overline{CD}^2$. And extracting the square Roots, $PM : PN :: CB : CD$. And since we have always this Proportion, let the Parallel PMN be any where

* Art. 186. drawn; therefore * the Hyperbolick Spaces APM, APN, or $CPMB$, $CPND$, are to one another, as CB to CD. But the Triangles CPM, CPN,

CPN, are to one another, as their Bases PM, PN, (because they are situate between the same Parallels CD, PN), or as the Semiconjugate Diameters CB, CD. And consequently (*Fig.* 117.) $CB : CD :: CPM - APM : CPN - APN :: CAM : CAN$. Or else (*Fig.* 118.) $CB : CD :: CPMB - CPM : CPND - CPN :: CBM : CDN$. *W. W. D.*

COROLLARY.

216. If the two Semi-conjugate Diameters CA, CD, be equal to one another; then AN, or DN, will be an equilateral Hyperbola. And if we did know how to square the Hyperbolick Sectors CAN, or CDN, then should we have likewise the Quadrature of the Sectors CAM, or CBM, whose Bases are Parts (AM, or BM) of some other Hyperbola, and CD the Conjugate Diameter thereof equal to any given Magnitude: Because the Relation of the Hyperbolick Sectors CAM, CAN, or CDN, CBM, being express'd by the right Lines CD, CB, is given. Therefore, if we could get the Quadrature of the equilateral Hyperbola, we should have likewise the Quadrature of any other Hyperbola: Just as the Quadrature of all Ellipses might be had by having * the Quadrature of the Circle. *Art. 212.

PROPOSITION VII.

Theorem.

217. *If the Parts* CK, CL *be assum'd in one Asymptote* (CN) *of any Hyperbola* EBDF, *having the same Ratio, as any two other Parts* CG, CH, *of the same Asymptote; and if the Lines* GF, HD, KB, LE, *be drawn parallel to the other Asymptote* CP, *meeting the Hyperbola in the Points* F, D, B, E; *and lastly, if the Semidiameters* CF, CD, CB, CE, *be drawn; I say, the two Hyperbolick Sectors* CBE, CDF, *will be equal to one another.* FIG. 119.

Draw the two right Lines BD, EF, meeting the Asymptotes in the Points M, O, N, P; then because KB, HD, and LE, GF are parallel, we have the two following Proportions, $MB : MK :: DO : CH$, and $NE : NL :: FP : CG$. And therefore, $MK = CH$, and $NL = CG$, because * $MB = DO$, and $NE = FP$. But (by the Hypothesis) CG, or $LN : CH$ or $KM :: CK : CL :: LE :$ * KB. and therefore $LN : LE :: KM : KB$. And consequently the Lines NE, MB, that is, EF, BD, are parallel. Therefore the Hyperbolick Sectors CBE, CDF, are * equal to one another. *W. W. D.* *Art. 95. *Art. 100. *Art. 209.

COROLLARY I.

218. IF the Parts CK, CL, of one Asymptote CN, be in the same Proportion, as any two Parts CS, CT, of the other Asymptote CP; and if the Lines KB, LE, and SD, TF, be drawn parallel to the said Asymptotes; then it is plain, that the Hyperbolick Sectors CDF, CBE, shall be equal also to one another. For drawing FG,
*Art. 100. DH, parallel to the Asymptote CP, we have * this Proportion, *viz.*
* Hyp. $CG : CH :: HD$, or $CS : GF$, or CT* $:: CK : CL$. Therefore, *&c.*

COROLLARY II.

219. IF CK be taken in the aforesaid Asymptote CN, equal to a third Proportional to any two Parts CG, CH, of the same; then we can prove, after the same manner as in the Theorem, that the Line BF is parallel to the Tangent passing through the Point D;
*Art. 210. and so * the Hyperbolick Sectors CFD, CDB, are equal to one another. Therefore, if any Number of Parts, CG, CH, CK, CL, &c. be taken in a continued Geometrical Progression in one Asymptote CN, and the right Lines GF, HD, KB, LE, &c. be drawn from them parallel to the other Asymptote; then the Hyperbolick Sectors CFD, CDB, CBE, &c. are every of them equal to one another.

COROLLARY. III.

220. HENCE, if CH be the first of two mean Proportionals between CG, CL. And if the right Lines GF, HD, LE, are parallel to the Asymptote CP; then the Sector CDF, to the Sector CFE, will be as 1 to 3. And if CH be the first of three mean Proportionals between CG, CL; the Sector CDF, will be to the Sector CFE, as 1 to 4. And universally, if m denotes any whole Number, and CH be the first of as many mean Proportionals between CG, CL, as the Number $m-1$ contains Units; then the Sector CDF will be to the Sector CFE, as 1 is to the Number m.

SCHOLIUM.

221. HENCE we may give a very exact Idea of those Numbers in Arithmetick, call'd Logarithms; and likewise shew their great Use in facilitating extremely Arithmetical Operations, wherein large Numbers are concern'd.

Let CG express Unity, CL the Number 10, and suppose the Hyperbolick Sector CFE to be divided into 10000000000 equal Parts; then

then if there be a Table divided into two Columns, in the first of which are orderly contain'd all the natural Numbers 1, 2, 3, 4, 5, 6, *&c.* and in the other, artificial Numbers standing against them, expressing the Number of Parts that the Hyperbolick Sector CDF does contain, with respect to the Number of Parts contain'd in the Sector CEF; then the artificial Numbers are call'd the *Logarithms* of the natural Numbers answering to them. This being premis'd,

1. If any two natural Numbers CH, CK, be propos'd to be multiply'd by one another, you need but take their Logarithms in the Table express'd by the Sectors CFD, CFB; and then adding the two Logarithms together, you will have the Logarithm expressing the Sector CFE, against which in the first Column stands the natural Number CL, which is the Product of the Multiplication of the two Numbers CH, CK.

2. If it be propos'd to divide the Number CL by the Number CK, you need only substract the Logarithm (CFB) of the Divisor CK, from the Logarithm (CFE) of the Dividend, and the Remainder CBE, or CFD, will be the Logarithm of the Quotient CH.

3. If it be propos'd to extract any Root of the Number CL, for Example, the Cubick; then you need only divide the Logarithm (CFE) thereof into three equal Parts, and you will have the Logarithm CFD, against which stands the Number CH, which is the Cube Root sought.

All this follows from the Equality of the Hyperbolick Sectors CFD, CBE, and CFD, CDB, CBE, &c. when $CG : CH :: CK :: CL$, and $CG : CH :: CH : CK :: CK : CL ::$ &c. Therefore by such a Table it is manifest, that Arithmetical Operations in great Numbers may be vastly abbreviated, and so the Logarithms are of great Use in Trigonometry and Astronomy, *&c.*

Because the Relation of the Hyperbolick Sectors CFD, CFB, *&c.* to the Sector CFE, cannot be express'd exactly in Numbers; therefore that Relation is express'd in Numbers nearly; and by means of these Numbers, (call'd *artificial* ones) and the natural Numbers set against them is a Table of Logarithms compos'd, which has all the Properties we have here explain'd. Now according to the Supposition, that the Sector CFE being the Logarithm of CL (10) contains 10000000000 equal Parts, we shall find that the Parallelogram $CGFT$, contains more than 4342944818 of those Parts, and less than 4342944819. Therefore any Hyperbolick Sector CBF, is to the Parallelogram $CGFT$, nearly as the Logarithm of the Number CK (taken to ten Places of Figures besides the Characteristick) is to the Number 4342944819.

PRO-

PROPOSITION VIII.

Theorem.

FIG. 120. 222. *IF in each of the Asymptotes there be taken the Parts* CG, CL, *and* CR, CS, *being such that* $\sqrt[m]{CG} : \sqrt[m]{CL} :: \sqrt[n]{CR} : \sqrt[n]{CS}$; *and if the right Lines* GF, LE, RT, SV, *be drawn parallel to the Asymptotes; I say, the Sector* CFE *will be to the Sector* CTV, *as* m *is to* n: *The Letters* m, n, *denoting any whole Numbers at pleasure.*

For if you make $\sqrt[m]{CG} : \sqrt[m]{CL} :: CG : CH$, and $\sqrt[n]{CR} : \sqrt[n]{CS} :: CR : CQ$. And if the right Lines HD, QN, be drawn parallel to
* Art. 218. the Asymptotes; then it is manifest, * that the Hyperbolick Sectors
* Hyp. CFD, CTN, shall be equal to one another; because * $CG : CH :: CR : CQ$. But by the Nature of Geometrical Progressions, the Line CH will be the first of as many mean Proportionals between CG and CL, as the Number $m-i$ contains Units, and the Line CQ, the first of as many mean Proportionals between CR and CS, as the Number
* Art. 220. $n-i$ contains Units. Therefore * $CFE : CFD :: m : i$, and CTN, or $CFD :: CTV : i : n$. And consequently the Sector CFE is to the Sector CTV, in the Ratio compounded of m to i, and of i to n, that is, as the Number m is to the Number n. *W. W. D.*

COROLLARY.

223. HENCE, if any Hyperbolick Sector CFE be given, as also any Point T in the Hyperbola; and if it be requir'd to find some other Point V in the said Section, so that the Sector CFE be to the Sector CTV, as m is to n; then you must assume CS, such that $\sqrt[m]{CG} : \sqrt[m]{CL} :: \sqrt[n]{CR} : \sqrt[n]{CS}$, or (which comes to the same) $\sqrt[m]{\overline{CG}^n} : \sqrt[m]{\overline{CL}^n} :: CR : CS$; that is, you must take $CS = CR \times \sqrt[m]{\frac{\overline{CL}^n}{\overline{CG}^n}}$.

PROPOSITION IX.

Theorem.

FIG. 121. 224. *IF the right Lines* BK, FG, *be drawn through the Extremities* B, F, *of any Hyperbolick Sector* CBF, *parallel to one Asymptote* CS, *and terminating in the other* CL; *I say, the Hyperbolick Sector* CBF *is* equal

equal to the Hyperbolick Space BKGF, *contain'd between the Parallels* BK, FG, *the Part* GK *of the Asymptote* CL, *and the Portion* BF *of the Hyperbola.*

For if the Triangle *CGA* be taken from the equal * Triangles *CKB*, *CGF*, (the Point *A* being the Intersection of the two right Lines *FG*, *CB*,) and if to the two Remainders *BKGA*, *CAF*, there be added the Hyperbolick Space *BAF*; then on one Side we shall have the Space *BKGF*, equal to the Sector *CBF* on the other. *W. W. D.* * *Art.* 99.

Corollary I.

225. If the Lines *BQ*, *FO*, be drawn parallel to the Asymptote *CL*, and terminating in the Asymptote *CS*; then we can prove by the like Reason, that *CBF* is equal to the Hyperbolick Space *BQOF*; from whence it appears, that the Spaces or Hyperbolick Trapezia *BKGF*, *BQOF*, are equal to one another.

Corollary II.

226. Hence, whatever has been demonstrated in the 217th, 218th, 219th, 220th, 221st, 222d, and 233d Articles of Hyperbolick Sectors, extends to the aforesaid Trapazia, because these are equal to the Sectors.

Proposition X.

Theorem.

227. *If there be two Hyperbola's* BMF, HND, *having the same Asymptotes* CL, CS; *and if through any two Points* G, K, *in one Asymptote, there be drawn the right Lines* GDF, KHB, *parallel to the other; I say, the Hyperbolick Space* HKGD *to the Hyperbolick Space* BKGF, *is as the Power of the Hyperbola* HND, *to the Power of the Hyperbola* BMF. Fig. 128.

For through any Point *P* in the Part *GK*, draw a Parallel to *GD* or *KH*, meeting the Hyperbola *BMF*, in the Point *M*, and the Hyperbola *HND*, in the Point *N*; and call the Powers of the Hyperbola's *HND*, *BMF*, aa, bb; and the indeterminate Quantity *CP*, x; then will * PN be $=\frac{aa}{x}$, and $PM=\frac{bb}{x}$; and so $PN:PM::aa:bb$. And because this is so always, let the Point *P* be any where taken between *G* and *K*, therefore * the Hyperbolick Space *HKGD*, to the Hyperbolick Space *BKGF* is, as aa to bb. *W. W. D.* * *Art.* 101. * *Art.* 186.

Corol-

COROLLARY.

228. WHEN the Powers of the Hyperbola's HND, BMF, are to one another, as the Number m to the Number n; then in the Hyperbola HND we can find always an Hyperbolick Trapezium $RSVT$, equal to the Hyperbolick Trapezium $GKBF$, in the other Hyperbola BMF; the right Lines CG, CK, CR, being given.
*Art. 222, and 225. For it is manifest, * that the Trapezium $GKHD$, to the Trapezium $GKBF$, is as m to n; and so the whole Difficulty consists in finding the Trapezium $RSVT$, in the Hyperbola HND, which shall be to the Trapezium $GKHD$, as the Number n to m. And this may be
*Art. 223, and 226. done * in taking CS such, that $\sqrt[m-n]{CG} : \sqrt[m-n]{CK} :: CR : CS$.

DEFINITIONS.

4.

FIG. 123. If there be an indefinite right Line AC, whose Origin is the fix'd Point A; and if there be a Curve AMB such, that a right Line MP being drawn from any Point P in the same, making a given Angle APM with AC, and if, the indeterminate Quantities AP, PM, being call'd x, y, we have always $ax = yy$ (where the Letter a
* Art. 19. denotes a given Line); then it is plain, * that the Curve AMB is a Parabola, the right Line AC a Diameter, the right Line PM an Ordinate to that Diameter, and the given Line a the Parameter. But now, if the Nature of the Curve AMB be suppos'd to be express'd by this Equation $y^3 = aax$, or this $y^3 = axx$; then that Curve is call'd a *Cubick Parabola*, or a *Parabola of the third Degree*. Because the Power of one of the indeterminate Quantities x, y, arises to the third Degree. In like manner, if the Equation be $y^4 = a^3x$, or $y^4 = ax^3$; then the Curve AMB, is called a *Parabola of the fourth Degree*. Because one of the indeterminate Quantities, as y, arises to the fourth Degree, and so of others to Infinity.

5.

FIG. 124. If there be a right Line AC (as in the last Definition) and the fixed Point A be the Origin; and if there be a Curve Line BM such, that drawing the right Line MP from any Point M thereof, making a given
*Art. 11. Angle APM with AC, and calling AP, x, PM, y; we have always $= ax$ (where the Letter a denotes a Line given); then it is plain,

 * that

* that that Curve will be an Hyperbola, and the right Line AC one *Art. 101.
of its Asymptotes, the Line AD (parallel to PM) the other, and the Square aa the Power thereof. But if the Equation expressing the Nature of the Curve BM be $xxy = a^3$; then that Curve is called *a Cubick Hyperbola*, or an *Hyperbola of the third Degree*; because the Product xxy of the two indeterminate Quantities x and y, is of three Dimensions. Farther, if the Equation be $x^3y = a^4$: then the Curve BM is an *Hyperbola of the fourth Degree*. Because the Product x^3y is of four Dimensions: And so of others to Infinity.

Corollary I.

229. IF the Letter m denoting any whole Number, be the Exponent Fig. 123,
of the Power of the indeterminate Quantity AP (x); and if 124.
likewise the Letter n be the Exponent of the other indeterminate Quantity PM (y); then it is evident, that the Equation $y^n = x^m \times a^{n-m}$; or (for Brevity's Sake, making the given Quantity $a = 1$) $y^n = x^m$, expresses the Nature of Parabola's of all the Degrees to Infinity. In like manner the Equation $x^m y^n = a^{m+n}$, or (making $a = 1$) $x^m y^n = 1$, expresses generally the Nature of all Kinds of Hyperbola's.

Corollary II.

230. IF any indefinite right Line AD be drawn thro' A the fix'd Origin of the Line AC, parallel to PM; and if MK be drawn parallel to AC, meeting AD in the Point K, and the indeterminate Quantities AR, KM, be call'd x, y; then it is evident, that the indeterminate Quantity x, which express'd before the Line AP, or MK, will now be y; and contrariwise y, which express'd PM or AK, will now be x. From whence it follows,

1. If the Curve AMB be a common Parabola, then the Equation Fig. 123.
thereof will be $yy = ax$, or $xx = ay$, according as the Points of the Parabola regard the Points of the Line AC or AD; if that Curve be a Cubick Parabola, then the Equation expressing the Nature thereof will be $y^3 = aax$, when the Points thereof regard the Line AC, or $x^3 = aay$, when the Points thereof regard those of the Line AD; and $y^n = x^m a^{n-m}$, or $x^n = y^m a^{n-m}$, will express in general the Nature of the Parabola AMB, according as the same regards the right Line AC or AD, where n is supposed to exceed m.

2. The common Hyperbola is always express'd by this Equation Fig. 124.
$xy = aa$, whether it regards the Line AC or AD, the Cubick Hyperbola by this $xxy = a^3$, when it respects AC, and by this $xyy = a^3$, when it regards the other Line AD. And lastly, $x^m y^n = a^{m+n}$

 or

or $x^n y^m = a^{m+n}$, will express in general the Nature of the Hyperbola, according as the Points thereof have regard to those of the Line AC, or of AD.

Corollary III.

231. HENCE it is manifest, that there are two Cubick Parabola's, one of which is express'd by the Equation $y^3 = aax$, or $x^3 = aay$, and the other by $y^3 = axx$, or $x^3 = ayy$; whereas there is but one Cubick Hyperbola $xxy = a^3$, or $xyy = a^3$. For the indeterminate Quantities x and y, can be combined but four ways for expressing the Nature of the Cubick Parabola, and but two for expressing the Nature of the Cubick Hyperbola. And because the four first Equations appertain to two different Curves, and the two last to the same; therefore, &c. And after this manner may be found the Number of Parabola's or Hyperbola's of the fourth Degree, fifth Degree, &c.

Corollary IV.

Fig. 124. 232. THE indefinite right Lines AC, AD, may not only be Asymptotes of the common Hyperbola, but moreover, Asymptotes of any other Hyperbola of whatsoever Degree. For let the general Equation, expressing the Nature of any Hyperbola, be $x^m y^n = a^{m+n}$, or $y^n = \frac{a^{m+n}}{x^m}$ $(AP = x,\ PM = y)$ when the Points of it are referr'd to those of the Line AC; then it is manifest, that the more $AP\ (x)$ increases, the more, on the contrary, will y^n, and consequently $PM\ (y)$ diminish; so that when x is infinitely great, $PM\ (y)$ will become nothing; that is, the Hyperbola BM, and the Line AC, being both infinitely produced, do continually approach nearer and nearer to one another, until they meet one another at an infinite Distance; and so that Line will be an Asymptote. Now if the Points of the same Hyperbola be referr'd to those of the Line AD, then we shall have $x^n y^m = a^{m+n}$, or $y^m = \frac{a^{m+n}}{x^n}$. $(AK = x,\ KM = y)$; from whence it follows, that the more $AK\ (x)$ increases, the more doth $KM\ (y)$ diminish, and so the Line AD is an Asymptote also of the same Hyperbola.

PROPOSITION XI.

Problem.

Fig. 123. 233. *IT is requir'd to draw the Tangent* MT *to the Point* M *given in the Cubick Parabola* AMB, *whose Nature is expressed by the Equation* $= axx$.

Suppose

Suppose the Arc MN to be infinitely small, and draw NQ, parallel to PM, and MR parallel to AC; then the small Triangle MRN, will be similar to the great one TPM; because the small Arc MN may be taken * for a Part of the Tangent TM produced. This being laid down, call the Sub-tangent (TP) sought, s; the small right Line PQ, or MR, e; and then we shall have $RN = \frac{ey}{s}$; because the Triangles TPM, MRN, are similar. Now, if the Cube of QN $(y + \frac{ey}{s})$ be put for y^3 in the Equation $y^3 = axx$, expressing the Nature of the Curve AMB, and the Square of AQ $(x + e)$ for xx; then we shall form this Equation $y^3 + \frac{3ey^3}{s} + \frac{3eey^3}{ss} + \frac{e^3y^3}{s^3} = axx + 2eax + eea$, which expresses the Relation of AQ to QN. And if the Equation $y^3 = axx$ be taken from this Equation, and the remaining Equation be divided by e; then there will arise $\frac{3y^3}{s} + \frac{3ey^3}{ss} + \frac{eey^3}{s^3} = 2ax + ea$; and striking out all the Terms wherein e happens, because PQ (e) being infinitely small, those Terms are infinitely little in respect of the others; and then we shall get $\frac{3y^3}{s} = 2ax$; and therefore $PT(s) = \frac{3y^3}{2ax} = \frac{3}{2}x$, in substituting y^3 for axx the Value thereof. *W. W. D.*

* *Art.* 189.

SCHOLIUM.

234. IF Regard be had to the aforegoing Process, it will appear by substituting, instead of the Power of y, a like Power of $y + \frac{ey}{s}$, that there is only Occasion for the two first Terms of that Power; for all the other Terms being multiply'd by the Powers of e, have either e, or Powers of e, in the last Equation found at the End of the Operation; and so these Terms must consequently vanish. Understand the same in putting for the Power of x, a like Power of $x + e$. But if all the Powers of the Binomial $x + e$, be successively formed; then $x^2 + 2ex$ will be the two first Terms of the second Power; $x^3 + 3exx$, the two first Terms of the third; $x^4 + 4ex^3$, the two first Terms of the fourth; $x^5 + 5ex^4$, the first Terms of the fifth; and so on. So that the two first Terms of any Power (m) of $x + e$, will be $x^m + mex^{m-1}$. After the same manner it will be found, that the two first Terms of any Power (n) of the Binomial $y + \frac{ey}{s}$, will be $y^n + \frac{ney^n}{s}$.

COROLLARY.

235. HENCE we have a general Expreſſion for the Sub-tangent $PT(s)$ of all Kinds of Parabola's, by means of the general Equation $y^n = x^m a^{n-m}$, or (making $a = 1$) $y^n = x^m$ expreſſing the Nature of all Sorts of Parabola's.

For ſubſtitute the two firſt Terms of the Power (n) of $y + \frac{ey}{s}$, that is, $y^n + \frac{ney^n}{s}$, for y^n in the general Equation $y^n = x^m$; and the two firſt Terms of the Power (m) of $x + e$, that is, $x^m + mex^{m-1}$ for x^m; and then we ſhall get $y^n + \frac{ney^n}{s} = x^m + mex^{m-1}$. And ſubſtracting the firſt Equation from this, and then dividing by e, we ſhall have $\frac{ny^n}{s} = mx^{m-1}$; and therefore $s = \frac{ny^n}{mx^{m-1}} = \frac{n}{m}x$, by putting x^m for y^n.

PROPOSITION XII.

Problem.

FIG. 124. 236. TO *draw Tangents to all Kinds of Hyperbola's.*

The ſame Preparation being made, as in the aforegoing Propoſition, ſubſtitute the two firſt Terms of the Power (m) of AQ $(x + e)$ that is, $x^m + mex^{m-1}$ for x^m, in the general Equation $x^m y^n = a^{m+n}$ expreſſing the Relation of $AP(x)$ to $PM(y)$: and the two firſt Terms of the Power (n) of $QN\left(y - \frac{ey}{s}\right)$ that is, $y^n - \frac{ney^n}{s}$ for y^n: and ſo by Multiplication we ſhall get this Equation $x^m y^n + mey^n x^{m-1} - \frac{ney^n x^m}{s} - \frac{mneey^n x^{m-1}}{s} = a^{m+n}$ expreſſing the Relation of AQ to QN. And ſubtracting the firſt Equation from this, and then dividing by ey^n, and there will come out $mx^{m-1} - \frac{nx^m}{s} - \frac{mnex^{m-1}}{s} = 0$. And ſtriking out the Term $\frac{mnex^{m-1}}{s}$, being incomparably ſmall with reſpect to the two others, becauſe the infinitely ſmall Line $PQ(e)$ multiplies the ſame, and we ſhall get $PT(s) = \frac{nx^m}{mx^{m-1}} = \frac{n}{m}x$.

COROL-

Corollary.

237. HENCE, if it be requir'd to draw a Tangent MT to the given Point M in any Parabola or Hyperbola of whatsoever Degree, the Nature of the Parabola being express'd by $y^n = x^m a^{-m}$, and of the Hyperbola by $x^m y^n = a^{m \times n}$: you need only assume the Sub-tangent $PT = \frac{n}{m} AP$, on the same Side the Point A as P is, when the Curve is a Parabola, and on the other Side when the same is an Hyperbola. Fig. 123, 124.

PROPOSITION XIII.

Theorem.

238. *LET there be a Parabola* AMB *of any Kind, whose Nature is express'd by the Equation* $y^n = x^m a^{-m}$: *and suppose the right Line* BC *to be drawn from any Point* B *in the same, making a given Angle* (ACB) *with* AC : *and compleat the Parallelogram* ACBD. *I say, the circumscrib'd Parallelogram* ACBD *is to the Parabolick Space* ACBMA, *contained under the straight Lines* AC, CB, *and the Part* AMB *of the Parabola, as* m + n *is to* n. Fig. 125.

We are to prove, that ACBD : ACBMA :: m [illegible]

Suppose an Arc (MN) of the Parabola AMB to be infinitely small, or (as some please to speak) indefinitely small, that is, so very small as to be less than any given Part of that Parabola; and draw the right Lines MP, NQ, parallel to BC; and MK, NL, parallel to AC, forming the small Parallelogram $MRNS$: Also draw the Tangent MT, meeting the Diameter AC in the Point T, from which draw a right Line parallel to CB, meeting the Lines MK, NL in the Points F, G. This being done, the small Arc MN may be taken * for one of the infinitely small Sides of the Polygon, making up the Part or Portion (AMB) of the Parabola, and the Tangent MT, for that small Side continu'd out; so that we have two right-lin'd Triangles NRM, MPT, which are similar: Whence NR or $MS : RM :: MP : PT$ or MF. And therefore the Parallelogram $PMRQ$ is equal to the Parallelogram $FMSG$; because the Angles PMR, FMS, are equal, and the Sides about these Angles are reciprocally proportional. *Art. 189.

But * MF or $PT = \frac{n}{m} AP$ or $\frac{n}{m} MK$. Therefore also the Parallelogram $FMSG$, or its equal $PMRQ = \frac{n}{m} KMSL$. And since this is always so, let the small Arc MN be taken any where on the Portion *Art. 137.

 of

of the Parabola; therefore the Sum of all the small Parallelograms
* Art. 184. $PMRQ$, that is, the * Trilineal Parabolick Figure $ACBMA$ is $= \frac{n}{m} ADBMA$ the Sum of all the small Parallelograms $\frac{n}{m} KMSL$. Whence $ADBMA : ACBMA :: m : n$. And consequently $ADBMA + ACBMA$, or $ACBD : ACBMA :: m + n : n$. *W. W. D.*

COROLLARY.

239. HENCE it is evident, that the Trilineal Parabolick Figure APM to the circumscrib'd Parallelogram $APMK$, is as n to $m + n$; and so the Parabolick Trapezium $MPCB$ is $= \frac{n}{m+n} ABCD - \frac{n}{m+n} APMK$; because $ACBMA = \frac{n}{m+n} ACBD$, and $APM = \frac{n}{m+n} APMK$.

PROPOSITION XIV.

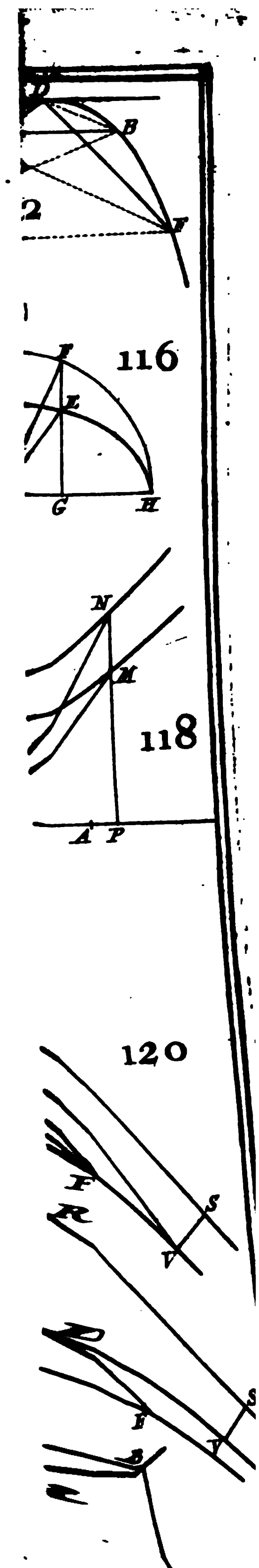
Theorem.

FIG. 126. 240. *LET there be an Hyperbola* BMO, *of any Kind, whose Nature is express'd by the Equation* $x^m y^n = a^{m+n}$; *and suppose the right Line* BC *to be drawn from any Point* B *therein parallel to one of the Asymptotes* AD, *and terminating in the other* C, *and compleat the Parallelogram* ACBD; *I say, the said inscrib'd Parallelogram* ACBD *is to the Hyperbolick Space* ECBMO, *contain'd under the determinate right Line* BC, *the right Line* CE *indefinitely produced towards* E, *and the Part* BOM *of the Hyperbola infinitely produced towards* O, *as* m—n *is to* n.

We are to prove, that ACBD : ECBMO :: m—n : n.

The same Preparation being made as in the last Proposition, we prove, as we have done there, that the small Parallelogram $PMRQ$ is $= \frac{n}{m} KMSL$. And because this is always so, let the infinitely small Arc MN be suppos'd to be taken in any Part (BMO) of the Hyperbola; therefore the Sum of all the little Parallelograms
* Art. 184. $PMRQ$, that is, the Space * $ECBMO$ is $= \frac{n}{m} EADBMO$, the Sum of all the small Parallelograms $\frac{n}{m} KMSL$. Whence we have $EADBMO : ECBMO :: m : n$; and therefore $EADBMO - ECBMO$, or $ACBD : ECBMO :: m - n : n$. *W. W. D.*

COROL-


D
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116
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A
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120
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B

COROLLARY I.

41. HENCE it is manifest, that the Hyperbolick Trapezium $CPMB$ is $= \frac{n}{m-n} ACBD - \frac{n}{m-n} APMK$; because $CBMO$ is $= \frac{n}{m-n} ACBD$, and for the same Reason the Space PMO is $= \frac{n}{m-n} APMK$.

COROLLARY II.

42. HENCE, 1. If m be greater than n; then the Relation of the inscrib'd Parallelogram $ACBD$ to the Space $ECBMO$ indefinitely extended towards E, will be express'd always by positive Numbers; and so in this Case we have always the absolute Quadrature of that Space. 2. When $m=n$, which happens in the common Hyperbola; then the Relation of the Parallelogram $ACBD$ to the Hyperbolick Space $ECBMO$ is as o to 1; that is, the said Space is infinite in respect of the inscrib'd Parallelogram $ACBD$. 3. When m is less than n, then the inscrib'd Parallelogram $ACBD$ will be to the Hyperbolick Space $ECBMO$, as a negative Number is to a positive one; and therefore the Ratio of that Space to the Parallelogram $ACBD$, is (allowing the Expression) more than infinite. But it must be observ'd, in this Case, that the Hyperbolick Space included by the right Line DB, the Asymptote AD infinitely produced towards D, and the Hyperbola OMB infinitely produced towards B, will be to the inscribed Parallelogram $ACBD$, as m is to $n-m$, that is, the said Space is squarable: For if the indeterminate Quantities (x) be assumed on the Asymptote AD, instead of AC, then the Equation of the Hyperbola will become * $x^n y^m = a^{m+n}$. *Art. 230.

PROPOSITION XV.

Theorem.

243. IF *there be any Curve* AMB *within the right Angle* CAD, *and the right Line* MT, *touches the same in any Point* M *taken at pleasure; and if there be some other Curve* HFE, *within the Angle* DAH, *(adjacent to the Angle* CAD*) such, that, the Line* FM *being drawn from any Point* F *therein parallel to* AC, *meeting the Line* AD *in* K, *and the Curve* AMB *in* M, *the Line* AK *may be to* MT, *always as some constant Line* a *to* KF: *I say, if the Line* EB *be drawn through any Point* D *of the* FIG. 127.

the Line AD, *parallel to* AC, *and terminated by the two Curves, then the Space* ADEFH *will be equal to the Rectangle under the Curve* AMB, *and the constant Quantity* a.

We are to prove, that $ADEFH = AMB \times a$.

Suppose MN to be an infinitely small Arc taken any where in the Curve AMB, and draw the right Lines MF, NG, parallel to AC, meeting the right Line AD in the Points K, L, and the Curve HFE, in the Points F, G; also draw the right Lines FS, MR, parallel to AD, and produce RM till it meets AC in P. This being done, the two similar right-angled Triangles MPT, MRN, give this Proportion, *viz.* $MR : MN :: MP$ or $AK : MT :: a : KF$. And therefore $KF \times MR$, that is, the small Rectangle $FKLS$ is $= MN \times a$. And because this is so always, let MN be taken any where at pleasure in the Curve AMB; therefore the Sum of all the small Rectangles $KLSF$,
* *Art.* 184. that is, * the Space $ADEFH$, will be equal to the Sum of all the small Rectangles $MN \times a$, that is, equal to the Rectangle under the Curve AMB, and the constant Quantity a. *W.W.D.*

Corollary I.

244. HENCE the Rectangle under the Portion AM, and the constant Quantity a, is equal to the Space $AKFH$; and the Rectangle under the Portion MB, and the same Quantity a, is equal to the Space $KDEF$.

Corollary II.

245. IF the Curve AMB be supposed to be a Cubick Parabola, express'd by $y^3 = axx$, (AP being $= x$, and $PM = y$) then
* *Art.* 233. will * PT be $= \frac{3}{2} x$; and because the Triangle MPT is right-angled, the Hypothenuse MT will be $= \sqrt{yy + \frac{9}{4} xx}$. But by the Property of the Curve HFE, it must be as $MP\,(y) : MT\,(\sqrt{yy + \frac{9}{4} xx}) :: a$ KF. And so there arises $\overline{KF}^2 = aa + \frac{9aaxx}{4yy} = aa + \frac{9}{4} ay$, by substituting y^3 for axx. Whence the Curve HFE is a Parabola in this Case, the right Line AD being the Axis; the Point O falling on the other Side D with respect to A, (so that AO be $= \frac{4}{9} a$) being the Origin thereof and the Parameter $= \frac{9}{4} aa$; for by the Property of that
* *Art.* 19. Parabola, the Square of the Ordinate KF will be * equal to the Rectangle under KO, and the Parameter $\frac{9}{4} a$, that is, in analytick Terms, $\overline{KF}^2 = aa + \frac{9}{4} ay$. And because the parabolick Trapezia $ADEH$,
* *Art.* 239. $AKFH$, are * squarable; therefore we have the Rectification of the Curve AMB, or of any one of the Parts (AM) thereof.

If

If you have a mind to express the Value of the Part AM, you must observe, first, that AH is $= a$; because $\overline{AH}^2 = AO \times \frac{9}{4} a = aa$. Then calling the Tangent MT, t; and the Line AK or MP, y; and there arises $KF = \frac{at}{y}$, and the Parabolick Trapezium $FKAH$, or * $\frac{2}{3} FK \times KO - \frac{2}{3} HA \times AO$ will be $= \frac{2}{3} at + \frac{8aat}{27y} - \frac{8}{27} aa = AM \times a$. That is, (dividing by a) the Portion AM sought will be $= \frac{2}{3} t + \frac{8at}{27y} - \frac{8}{27} a$. From whence arises the following Construction. *Art. 239.

Draw the Tangent MT from any given Point M in the Cubick Parabola AMB, meeting the Line AK drawn through the Origin A of the Axis AC perpendicular to the same, in the Point Q, and one AK assume $AV = \frac{8}{27} a$; then if VC be drawn parallel to MT, meeting the Axis in C, and a Circle be describ'd about the Centre V, with the Radius VA, cutting VC in X; I say, the Part AM of the Cubick Parabola AMB, will be equal to the Sum of the two right Lines MQ, CX.

For because the Triangles TPM, TAQ, are similar, it is plain, that MQ is $= \frac{2}{3} MT\ (t)$, since $AP = \frac{2}{3} PT$; and because the Triangles MPT, VAC, are similar, we have this Proportion $MP\ (y) : MT\ (t) :: AV \left(\frac{8}{27} a\right) : VC = \frac{8at}{27y}$, and therefore $CX = \frac{8at}{27y} - \frac{8}{27} a$. Whence, &c.

PROPOSITION XVI.

Theorem.

246. *LET there be an Equilateral Hyperbola* EAF, *together with a Parabola* NCS; *and let* C *be the Centre of the Hyperbola, the Line* CA *half of its first Axis, and the Line* CA *produced beyond* C, *the Axis of the Parabola, having a Line the double of* CA *for the Parameter thereof, and the Point* C *for its Origin. Then if a right Line* NE *be drawn through any Point* N *in the Parabola* NCS, *parallel to* CA, *meeting the Hyperbola* EAF *in the Point* E, *and its second Axis* CL *in the Point* L. *I say, the Hyperbolick Space* CLEA, *included between the right Lines* AC, CL, LE, *and the Portion of the Hyperbola* EA *is equal to the Rectangle under the Portion of the Parabola* CN, *and the right Line* AC. FIG. 128.

Through any Point M in the Portion of the Parabola CN, draw MG perpendicular to the Tangent MT, as also MB parallel to CA, meeting the Hyperbola in B, and the second Axis CL in H; then the Lines MG, HB, will be equal to one another. For drawing the Ordinate MP, we have * $PG = CA$; and since the Triangle MPG is right-angled, the Square $\overline{MG}^2 = \overline{PM}^2 + \overline{PG}^2 = \overline{CH}^2 + \overline{CA}^2 =^* \overline{HB}^2$; because EAF is an equilateral Hyperbola; and so $MG = HB$. But the similar right-angled Triangles TPM, MPG, give this Proportion, MP or $CH : MT :: PG$ or $CA : MG$ or HB. Whence, * &c.

* Art. 24.
* Art. 127.
* Art. 143.

COROLLARY I.

247. HENCE it is manifest, that the Hyperbolick Trapezium $HLEB$ is equal to the Rectangle under the Portion of the Parabola MN, and CA the half of the Parameter of the Axis.

COROLLARY II.

248. IF any two Parallels BD, EF, be drawn in the equilateral Hyperboly EAF; and if the right Lines BM, EN, DR, FS, be drawn through their Extremities parallel to AC, meeting the second Axis of the Hyperbola in the Points H, L, K, O; then if the right Lines BE, DF be drawn, the Difference of the Rectangles $AC \times MN$, $AC \times RS$, will be equal to the Difference of the right-lin'd Trapezia $HLEB$, $KOFD$.

* Art. 247. For the Rectangle $AC \times MN$ is * equal to the Hyperbolick Trapezium $HLEB$; and consequently the Rectangle $AC \times MN$ *plus* the Hyperbolick Segment DF will be equal to the right-lin'd Trapezium $KOFD$. And so because the two Hyperbolick Segments EB, DF,
* Art. 204. are * equal to one another, the Difference of the Rectangles $AC \times MN$, $AC \times RS$, will be equal to the Difference of the right-lin'd Trapezia $HLEB$, $KOFD$. *W.W.D.*

COROLLARY. III.

249. THE same Things being premis'd as in the last Corollary; if it be made as $2AC : LH :: \overline{BH + LE} : m$; then it is plain, that the Rectangle $AC \times m = \frac{1}{2} LH \times \overline{BH + LE}$, that is, equal to the right-lin'd Trapezium $HLEB$. In like manner, if it be made

as

as $2AC : KO :: KD + FO : n$; then $AC \times n$ will be equal to the right-lin'd Trapezium $KOFD$. And consequently the Difference * of the Rectangles $AC \times MN$, $AC \times RS$, will be equal to the Difference of the Rectangles $AC \times m$, $AC \times n$; that is, dividing by AC, the Difference of the Parabolick Arcs MN, RS, will be equal to the Difference between the right Lines m, n. Whence it appears, that straight Lines may be found equal to the Difference of an infinite Number of Parabolick Arcs, such, as MN, RS. *Art. 248.

The End of the Fifth Book.

BOOK VI.

Of the Conick Sections consider'd in the Solid.

CHAP. I.

Of the Three Conick Sections in general.

DEFINITIONS.

1.

FIG. 129. IF there be any immoveable Point S assum'd without a Plane in which the Circle VXT is describ'd, and if a right Line SZ drawn through that Point, and infinitely produced both ways, moves quite round the Circumference of the Circle; then each of the two Superficies produced by the Motion of the indefinite right Line SZ, is call'd separately a *Conick Superficies*, and both of them conjunctly *opposite Conick Superficies*, or *only opposite Superficies*.

2.

The immoveable Point S, common to both the opposite Superficies, is call'd the *Vertex*.

3.

The Circle VXT, *the Base*.

4.

The Solid comprehended under the Base VXT, and that Part of the Conick Superficies between the Base and the Vertex S, is called a *Cone*.

5.

The Line SX drawn from the Vertex S to any Point X in the Base, is a *Side* of the Cone.

6. The

6.

The Line *SO* drawn from the Vertex *S*, through *O* the Centre of the Base, is call'd the *Axis* of the Cone.

7.

If the Axis be perpendicular to the Plane of the Base, then the Cone is call'd a *right Cone*; and if the Axis be not, then the Cone is call'd a *Scalene Cone.*

8.

If a Conick Superficies be cut by a Plane (*FAG*) not passing thro' the Vertex *S*, or parallel to the Plane of the Base *VXT*; the Curve *FAG*, form'd by the Concurrence of that Plane and the Conick Superficies, is a *Conick Section.* Fig. 130, 131, 132.

9.

If a Plane (*SDE*) be drawn through the Vertex (*S*) of a Cone, parallel to the Plane of a Conick Section; the indefinite right Line *DE*, formed by the Concurrence of that Plane, and the Base of the Cone, is call'd a *Directrix.*

10.

A Conick Section (*FAG*) is call'd a *Parabola*, when the Directrix *DE* touches the Circular Base of the Cone; an *Ellipsis*, when the same falls quite without that Base; and an *Hyperbola*, when it falls within [or cuts] that Base.

But in this last Case, if the Plane of the Section be continued, the same will meet and cut the opposite Conick Superficies; and the Curve *KMH* form'd thereby, is call'd an opposite Hyperbola with regard to the former Hyperbola *FAG*; and both of them together are call'd *opposite Hyperbola's*, or *opposite Sections.* Fig. 132.

11.

If any straight Line, in the Plane of a Conick Section, meets that Section in one Point only, and being both ways infinitely continued, does not cut or fall within the Section; then that Line is call'd a *Tangent*; and the Point wherein it meets the Section, is named the Point of *Contact.* Fig. 130, 131, 132.

Corol.

COROLLARY I.

FIG. 130. 250. IN the Parabola, all the Sides of the Cone being produced indefinitely will necessarily meet the Plane thereof, except the Side *SD* only, which is drawn from the Vertex *S* through the Point *D*, wherein the Directrix touches the Base; because there can be only that Side in the Plane *SDE*, parallel to the Plane of the Section, and all the other Sides cut the same in the Point *S*. From whence it appears, that the Parabola is a Curve of an infinite Extension, which doth not return into itself.

COROLLARY II.

FIG. 131. 251. IN an Ellipsis, all the Sides of the Cone being produced (if necessary) will meet the Plane thereof; because all the Sides of the Cone meet the Plane *SDE* parallel thereto in the Point *S*. From whence it appears, that the Ellipsis includes a Space, and returns into itself.

COROLLARY III.

FIG. 132. 252. IN the opposite Hyperbola's, all the Sides of the Cone, except only *SD*, *SE*, which are drawn from the Vertex *S* to the Points *D*, *E*, wherein the Directrix cuts the Base, being both ways infinitely produced, must necessarily meet their Plane; because there can be but these two Sides that fall in the Plane *SDE*, which is parallel to the Plane of the Hyperbola's, and all the other Sides cut that Plane in the Point *S*. Farther, the Sides of the Part of the Cone *SDVE* do form the Points of the Hyperbola *FAG*, and the Sides of the Parts *SDTE* being produced beyond the Vertex *S*, do form the Points of the opposite Hyperbola *KMH*. From whence it appears, that each of the opposite Sections are infinite, and do not return into themselves, no more than the Parabola.

PROPOSITION I.

Theorem.

FIG. 132. 253. *IF two opposite Superficies be cut by a Plane* Sam, *passing through the Vertex* S; *I say, the common Sections of that Plane, and the opposite Superficies, will be two straight Lines* Sa, Sm, *both ways indefinitely extended from* S.

For

For let $a m$ be the common Section of the cutting Plane, and the Plane of the Base; then it is plain, that $a m$ will cut the Base in two Points a, m; because the Plane $S a m$ falls within the Conick Surface. And so if the Sides $S a$, $S m$, be drawn, and produced indefinitely both ways from the Vertex S; then it is manifest by the Generation of opposite Superficies, that those Sides will be the two common Sections of the said Superficies, and the cutting Plane $S a m$. *W. W. D.*

COROLLARY I.

254. BEcause that Part of the Line $a m$, joining the two Points a, m, falls within the Base, and all the rest of that Line without the same; therefore, if the Plane $S a m$ be suppos'd indefinitely extended every way, that Part thereof included in the Angle $a S m$, as also in the opposite vertical Angle, will fall within the two opposite Superficies; and all the rest of that plane will fall without the said two Superficies.

COROLLARY II.

255. HENCE if any two Points (A, M) of a Conick Section be join'd by a straight Line, that Line will fall within the Section, and being both ways produced, will fall without the same. For drawing the Sides $S a$, $S m$, from the Vertex S through the Points A, M, and drawing a Plane through these Sides; it is manifest, that the Line $A M$ falls in that Part of this Plane included in the Angle $a S m$, and all the rest of that Line will be in the Part of the Plane falling in the adjoining Angles. FIG. 130.

COROLLARY III.

256. IF through the Vertex of the Cone S, there be drawn a right Line parallel to the Line $A M$, terminating in any Conick Section; then it is evident by the last Corollary, that that Line $S H$ will fall in one of the Angles adjoining to the Angle $a S m$, that is, it will fall without the Conick Superficies; and so the same will meet the Plane of the Base in some Point without the Circumference, or else will be parallel to the Base.

COROLLARY IV.

257. IT is manifest (by *Corol.* I.) that if any two Points (A, M) of two opposite Hyperbola's be join'd by a right Line, then that Line will fall within the said Hyperbola's; and if the same be both FIG. 132.

 ways

ways continu'd, it will fall without them: for drawing the Sides Sa, Sm, thro' the Vertex (S) and the Points A, M, and drawing a Plane through those Points and the Vertex both ways indefinitely extended from S; then it is evident, that that Part of this Plane included in the Angle ASM, wherein the Line AM falls, is contain'd between those two Superficies; and likewise that the Part of the said Plane included between the two adjoining Angles, (wherein are the Continuations of the Line AM) falls within those two Superficies. And because the Line AM is the common Section of the Plane Sam, and that of the two opposite Hyperbola's; therefore, &c.

COROLLARY V.

258. IT is manifest by the 2d and 4th Corollaries, that a right Line can meet a Conick Section, or the opposite Sections, in two Points only, and not more.

PROPOSITION II.

Theorem.

FIG. 129. 259. *IF either of the two opposite Superficies be cut by a Plane* ouxy *parallel to the Base* OVXY; *I say, the Section made by that Plane, and the Conick Superficies, is a Circle; and the Point* o, *wherein that Plane meets the Axis* SO, *(produced on the other Side the Vertex* S, *if necessary) is the Centre thereof.*

For if through any Point X in the Base, there be drawn the Radius XO to the Centre O, as also the Side XS from the Vertex S, meeting the Plane $ouxy$ in the Point x; then the Lines OX, ox, will be parallel to one another: Because they are the common Sections of the two parallel Planes $OVXT$, $ovxy$, and the Plane SOX (produced on the other Side the Vertex, if necessary) therefore the Triangles OSX, oSx, will be similar, and consequently this Proportion will be had always, *viz.* $SO : OX :: So : ox$. Now because the two first Terms of this Proportion are standing Quantities, therefore the fourth Term ox will be a standing Quantity, let the Point x be taken any where at pleasure; and consequently the Curve vxy is the Circumference of a Circle, and the Point o is the Centre.

COROLLARY.

260. HENCE the Base of a Cone may be put in any Place desir'd, according as it is found most proper so to do. And so when the Section is a Parabola or Hyperbola, it is placed commonly so as to

to cut the Section; but when the Section is an Ellipsis, it is sometimes placed so as to cut the same, and sometimes so that the Ellipsis be above it.

PROPOSITION III.

Theorem.

261. *IF through any Point* A, *taken in the Parabola* FAG, *there be drawn the indefinite right Line* AB *in the Plane thereof, within the Cone, parallel to the Side* SD, *and passing through the Point* D, *wherein the Directrix* DE *touches the Base; I say, that Line* AB *falls entirely within the Section, and being infinitely produced towards* B, *will never after meet the same.* Fig. 130.

For if the Plane *SAB* be drawn thro' the Vertex *(S)* of the Cone, and the Line *AB*; this Plane will form two Sides of the Cone by its Concurrence with the Superficies, one of which will be always the Line *SD*, because *AB* is parallel to it; and the other, the Line *Sa* which passes thro' the Point *A*. But the Plane *DSa*, contain'd between the Sides *SD*, *Sa*, infinitely produced towards *D*, *a*, does fall within the Conick Superficies: And consequently * the Line *AB*, which is always in that Plane, being parallel to the Side *SD*, shall fall wholly within the Parabola, and will never after meet it, tho' infinitely produced towards *B*. *Art. 254.

PROPOSITION IV.

Theorem.

262. *IF through any Point* A *taken in the Parabola* FAG, *there be drawn the right Line* AM *in the Plane thereof within the Cone, not being parallel to the Side* SD, *passing through the Point* D, *wherein the Directrix* DE *touches the Base; I say, that Line* AM *will meet the Parabola in some other Point* M. Fig. 130.

For if the Plane *SAM* be drawn through the Vertex *S*, and that Line; then this Plane will fall within the Conick Superficies, and will not pass through the Side *SD*; and therefore the same will form * the two Sides *Sa*, *Sm* of the Cone, one of which, as *Sa*, passes thro' the Point *A*; and the other *Sm*, is not parallel to the Plane of the Section, because *(Hyp.)* there is no Side but *SD*, which is parallel thereto. Therefore the Side *Sm* being produced (if necessary) will meet the Plane of the Parabola in some Point *M*, which the Line *AM*, formed by the Concurrence of the Plane *aSm*, and that of the Parabola, passes through. And it is manifest, that the Point *M* is *Art. 253.

in the Parabola *FAG*, because it is both in the Plane of the Section, and in the Superficies of the Cone. Whence, &c.

PROPOSITION V.

Problem.

FIG. 133, 134, 135. 263. TO *draw a Tangent, as* AF, *from a Point* A *given in a Conick Section.*

Through the Point *A*, and the Vertex (*S*) of the Cone, draw the right Line *SA*, meeting the Plane of the Base in the Point *a*, and draw the Tangent *Eaf* to the Point *a* in the Base; then the Line *AF*, made by the Concurrence of the Plane *SEaf* (produced beyond the Vertex, if necessary) and the Plane of the Section will be the Tangent sought.

For because the Tangent *Eaf* falls entirely without the Base, the Point *a* therein being excepted; therefore the Plane *SEaf*, indefinitely produced both ways from the Vertex *S*, will meet the opposite Superficies only in the Line *Sa* both ways indefinitely produced, and all the rest of that Plane falls quite without the Superficies. And consequently the Line *AF*, form'd by the Concurrence of the Plane *SEaf*, and the Plane of the Section, hath only the Point *A*, wherein the Line *Sa* meets the Plane of the Section, common to either of the opposite Superficies; and does fall wholly without the Section, that Point being only excepted. Therefore, &c.

COROLLARY I.

264. BEcause there can be but one Tangent *Eaf* drawn to the Point *a* in the Base of the Cone, therefore also there can be but one Tangent *AF* drawn to a Point (*A*) given in a Conick Section.

COROLLARY II.

265. FROM hence arises the Manner of drawing a Tangent *AF*, parallel to a right Line (*MN*) given in Position in the Plane of a Conick Section, or the opposite Sections. For if *SE* be drawn through the Vertex (*S*) of the Cone, parallel to *MN*, this Line will either meet the Directrix *DE* in some Point *E*, or else be parallel thereto; because this Line *SE* will be parallel to the Plane of the Section, and will fall consequently in the Plane *SDE*. Now if *SE* meets *DE* in the Point *E*, falling without the Circular Base of the Cone; draw the Tangent *Eaf* from the Point *E* to the Circle, and then it is manifest, that the common Section of the Plane *SEaf*, and the

the Plane of the Section, *viz.* AF is a Tangent, and will be parallel to the Line MN; because the two Sections (AF, SE,) of the parallel * Planes MAN, SED, made by the touching Plane $SEaf$, are parallel to one another, as well as * SE, MN. * *Hyp.*

Corollary III.

266. THE same Things being premis'd as in the last Corollary, it follows, Fig. 133.

1. That in the Parabola, the Problem is impossible, when the Line MN given in Position, is parallel to the Side SD passing through the Point D, wherein the Directrix DE touches the Base: For then, since the Point E falls in D, there can be no Tangent, but the Directrix DE drawn through that Point: And since the Plane passing thro' the Vertex, and the Directrix DE, is * parallel to the Plane of the Parabola, therefore there can be no Tangent form'd, because these two Planes cannot cut one another. But when the Line given in Position is not parallel to the Side SD, there may be drawn always one Tangent AF parallel to that Line, and no more: for the Point E falling then without the Base of the Cone, we can draw always Eaf, EDL, to that Base; the latter of which coinciding with the Directrix, is of no use for forming a Tangent in the Plane of the Section; but by means of the former Eaf, we may find always some Tangent AF by the Concurrence of the Plane $SEaf$, and the Plane of the Section, and that will be the Tangent sought. The same must be understood when the Line SE is parallel to the Directrix, for the Tangent Eaf will then become parallel to the Directrix; and so because there can be drawn but one Tangent parallel to the Directrix, since the Directrix itself touches the Base in the Point D, therefore, &c. * *Def.* 9.

2. In the Ellipsis, there can be drawn always two Tangents AF, BG, parallel to the Line MN given in Position, and consequently parallel to one another. For because all the Points of the Directrix DE fall without the Base; therefore there can be drawn always two Tangents Eaf, Ebg, from the Point E to that Base, not coinciding with the Directrix; and by means of these and the common Sections of the Planes $SEaf$, $SEbg$, and the Plane of the Section, the two Tangents AF, BG, will be form'd parallel to MN. The same must be understood, when the Line SE is parallel to the Directrix; for then, instead of the Tangents Eaf, Ebg, drawn from the Point E in the Directrix, you need only draw two Tangents parallel to the Directrix, which is possible always. Fig. 134.

3. In the opposite Sections, the Problem is impossible when the Point E falls within the Base of the Cone, because then there can no Tangent be drawn from that Point to the Base. But when the Point Fig. 135.

 E falls

E falls without the Base, then there may be found always two Tangents AF, BG, parallel to the Line MN given in Position; for because the Directrix cuts the Base, there can be drawn always two Tangents Eaf, Ebg, from the Point E to the Base, falling on both Sides the Directrix, by means of which the Intersection of the Planes $SEaf$, $SEbg$, and the Plane of the Section, will form the two Tangents AF, BG sought. The same must be understood, when the Line SE is parallel to the Directrix DE; for instead of the two Tangents Eaf, Ebg, you need only draw two Tangents parallel to the Directrix, which is always possible.

In this last Case it must be observ'd, that the parallel Tangents AF, BG, appertain always to the opposite Hyperbola's, and never to one and the same; for this is evident, because the Tangents (Eaf, Ebg) to the Base do fall necessarily on both Sides the Directrix DE.

COROLLARY IV.

267. IT follows from the last Corollary, (1.) That in the Parabola and Hyperbola, there cannot be two Tangents parallel to one another; and contrariwise, in the Ellipsis and opposite Sections, if any Tangent AF be given in Position, there can always be drawn another (BG) parallel to it.

(2.) If the Line MN given in Position, be bounded by a Conick Section; then, in the Parabola, there can be drawn always some Tangent AF parallel thereto; and in the Ellipsis and opposite Sections, two Tangents AF, BG; because the Line SE, drawn through
* Art. 256. the Vertex (S) parallel to MN, will * meet the Plane of the Base, either in some Point E without the Circumference, or be parallel thereto.

DEFINITIONS.

12.

FIG. 133. In a Parabola, if through any Point A you draw the right Line AB within the Cone parallel to the Side SD, passing through the Point D, wherein the Directrix DE touches the Base; the said Line AB is call'd a *Diameter*, and the Point A the *Origin* [or Vertex] thereof.

13.

FIG. 134, 135. In the Ellipsis or opposite Sections, any right Line AB, joining the Points of Contact of two parallel Tangents AF, BG, is call'd a *Diameter*; and the Points A, B are the Extremities thereof.

14. If

14.

If through any Point *P*, in any Diameter (*A B*) of a Conick Secti- FIG. 133,
on, there be drawn the right Line *M N*, (meeting the Section in the 134, 135.
Points *M, N,*) parallel to the Tangent, passing through the Origin (*A*) of that Diameter in the Parabola, and through either of the Extremities thereof in the other Sections; this Line *M N* is an *Ordinate* on both Sides to the Diameter *A B*, and either of its Parts *P M*, or *P N*, is an *Ordinate* to that Diameter.

15.

That Diameter, which is at right Angles to its Ordinates, is called an *Axis.*

COROLLARY.

268. IT is manifest by *Def.* 12. (1.) That all Diameters in a Parabola are parallel to one another, because they are all parallel to the Side of the Cone *S D*, drawn through the Point *D*, wherein the Directrix *D E* touches the Base. (2.) That there can be drawn but one Diameter through a Point given in the Plane of a Parabola, because there can only one Line be drawn through that Point parallel to the Side *SD*.

PROPOSITION VI.

Problem.

269. *ANY Diameter* AB *of a Conick Section, together with its Ordi-* FIG. 136,
nate PM, *being given, to describe the Section.* 137, 138.

Draw any Plane (the Plane of the Scheme *A P M* excepted) thro' the Ordinate *P M*, and draw the indefinite right Line *P a* through the Point *P* in that Plane perpendicular to *P M*, and describe a Circle about any Point *C* in that Line with the Radius *C M*. This being done,

1. When the Section is to be a Parabola, From one of the Points FIG. 136.
a, D, wherein the Circle cuts the Perpendicular *P A, viz. a,* draw the right Line *a A* through the Origin (*A*) of the Diameter *A B*, meeting the right Line *D S* drawn from the other Point *D*, in the Point *S*; then if a Conick Superficies be describ'd with the Point *S*, for the Vertex, and the Circle *D M a N* for a Base: I say, the Concurrence of that Superficies, and the Plane *APM*, will form the Parabola *MAN* sought. For if the Lines *D E, a f,* be drawn through the Extremities

of

of the Diameter Da parallel to PM; then it is manifest, that these Lines will be Tangents, because PM is * perpendicular to Da. But the Plane SDE passing through the Vertex of the Cone S, and the Tangent DE, is parallel to the Plane APM; since SD, DE, are * parallel to AP, PM: therefore * the Section MAN, form'd by the Plane APM in the Conick Superficies, will be a Parabola, and the Line AB, a Diameter thereof. Farther, the touching Plane Saf forms * the Tangent AF, in the Plane APM, which Tangent will be parallel to PM; because AF is the common Section of the two Planes Saf, APM, which pass through the Parallels af, PM; and consequently * the Line PM will be an Ordinate to the Diameter AB.

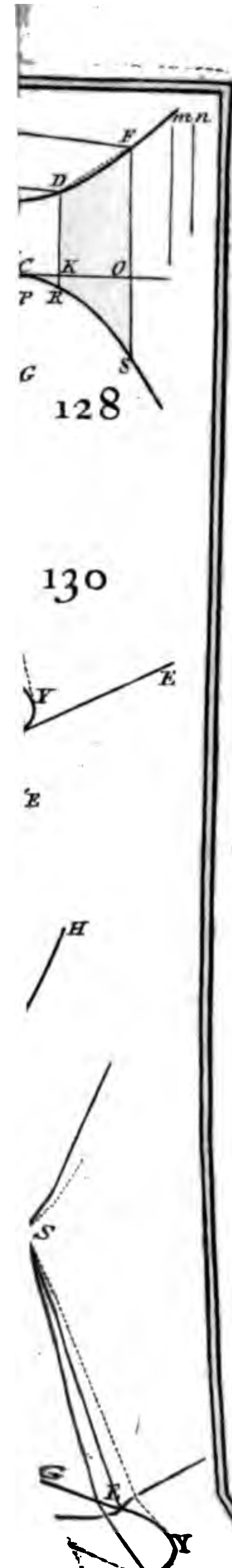
* Hyp.
* Hyp.
* Def. 10, and 12.
* Art. 263.
* Def. 14.

FIG. 137, 138. 2. When the Conick Section is to be an Ellipsis or Hyperbola. Draw the right Lines aA, bB, from the Points a, b, (wherein the indefinite Perpendicular Pa cuts the Circle) through the Extremities A, B, of the Diameter AB, meeting one another in the Point S. And then if a Cone be describ'd, with the Vertex S, and the Base $aMbN$; I say, the Plane APM will form the requir'd Section MAN, in the Superficies of the Cone. For if SD be drawn parallel to AB the Diameter of the Section, meeting ab the Diameter of the Base in D; and if DE, af, bg, be drawn through D, and the Extremities a, b, parallel to PM; then it is manifest, that the Plane SDE is parallel to the Plane APM, and so DE will be * the Directrix. But in the Ellipsis, the Point D falls on the Diameter ab produced without the Circle; since the Diameter (AB) of the Section, falls in the Angle aSb, form'd by the Sides of the Cone Sa, Sb; and contrariwise, in the Hyperbola the Point D falls within the Circle, because then the Diameter (AB) of the Section falls in the Angle aSB, adjoining to the Angle aSb. Therefore, by the 10th Definition, the Section MAN is an Ellipsis in the first Case, and an Hyperbola in the second. Farther, the Tangent AF, passing through the Extremity (A) of the Diameter AB, being the common Section of the touching Plane Saf, and the cutting Plane APM (passing thro' the Parallels af, PM,) will be parallel to PM. In like manner, the Tangent BG, being the common Section of the touching Plane Sbg, and the cutting Plane APM (passing thro' the Parallels bg, PM) will be parallel also to PM; therefore the Line AB is * a Diameter, and PM is an Ordinate to the same.

* Def. 9.
* Def. 13, and 14.

In the Ellipsis, it may happen that the Lines Aa, Bb, be parallel between themselves; in which Case, you may take any other Point in the Line ab, for the Centre of the Circle $aMbN$.

DEFI-



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P R
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130
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E
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S
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Y


5
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S
b


138

DEFINITION.

16.

Tangents DH, EK, be drawn thro' the two Points (D, E) the Directrix cuts the Base, when the Section is an Hyperbo- if two Planes SDH, SEK, be drawn through the Vertex S; two Tangents; the two indefinite right Lines CH, CK, the Concurrence of those Planes, and the Plane of the Hy- are call'd *Asymptotes.* FIG. 139.

COROLLARY.

ro' the Point of Contact D, there be drawn the Side DS, efinitely produced both ways from S; then it is evident, ane SDH will touch the opposite Superficies in that Line; the Points of the Tangent DH, except D, do fall without ference of the Base. But the Plane SDE passing through S, and the Directrix DE, being * parallel to the Plane of e Hyperbola's, the common Sections SD, CH, of those SD, CH, and the Plane SDH, will be parallel between ; and therefore the Asymptote CH will fall quite without en) the opposite Superficies, and consequently will both ways opposite Sections entirely without meeting them. The be prov'd of the other Asymptote CK, because the two es CH, CK, are form'd by the Planes SDH, SEK, falling des the same Conick Superficies, and the Superficies opposite refore all the Points of the Hyperbola FAG, are contain'd ngle HCK, and all the Points of the opposite Hyperbola gle vertically opposite thereto. Def. 9.

PROPOSITION VII.

Theorem.

the right Line B A *be drawn through any Point* B, *taken in one Asymptote* CK, *parallel to the other Asymptote* CH; *I say, this meet one of the opposite Hyperbola's in some Point* A *only, and efinitely produced, will be ever after within the same.* FIG. 139.

e the Lines BA, SD, are parallel to the same Line CH, they parallel between themselves; and so both of them will be in Plane cutting the opposite Superficies, since that Plane must gh one Side SD of the Cone, and make an Angle with the DH, which touches the Cone in that Side SD. Therefore

4 the

the Plane of the Parallels BA, SD, will form two Sides of the opposite Cones; one of which is the Side SD, and the other the Side Sa, which necessarily shall cut the Line BA in some Point A, because the same is situate in the Plane passing through the Parallels SD, AB, and does cut SD in S. Whence, because the Point A is in one of the Conick Superficies, and also in the Plane of the opposite Sections, this Point will be one Point of the Hyperbola. Moreover, because the Line BA being indefinitely produced towards A, falls wholly in the Plane DSa, contain'd between the Sides DS, Sa, when the Point A appertains to the Hyperbola FAG, and in the vertical Angle ASI, when the same appertains to the opposite Hyperbola; therefore the Line AB falls wholly within one of the Conick Superficies, and consequently also within the Hyperbola, which is the Section thereof.

COROLLARY.

272. HENCE it appears, that no Line can be drawn between the Hyperbola FAG, and its Asymptote CH parallel to that Asymptote. And because the Line BA divides the Hyperbola into two indefinite Parts or Portions, one of which falls necessarily wholly within the Space contain'd between the Parallels BA, CH; therefore the more CB diminishes, the more does the Point A advance in that Portion, even until CB becomes less than any given Magnitude; that is, if an Hyperbola and its Asymptote be indefinitely continu'd, they will approach always nearer and nearer to each other, till at last their Distance will become less than any given Magnitude, and yet they will * never meet.

* Art. 270.

PROPOSITION VIII.

Problem.

Fig. 140. 273. *THE Asymptotes* CH, CK, *of any Hyperbola* FAG, *together with any one Point* F *therein being given, to describe the Hyperbola.*

Draw any right Line HK through the given Point F, terminating in the Asymptotes, and draw any Plane (except the Plane HCK of the Scheme) thro' that Line, in which draw MN from the Point P, the Middle of HK perpendicular to HK; and about any Point O therein, as a Centre, and with the Radius OF, describe a Circle FMN. From the Points H, K, draw the two Tangents HD, KE to that Circle, and through the Points of Contact D, E, draw the Lines DS, SE, parallel to the Asymptotes CH, CK, meeting each other in the Point S. Then if a Conick Superficies be describ'd with the Vertex S, and

and the Base FMN; I say, the Concurrence of that Conick Surface and the Plane HCK, will form the Hyperbola FAG requir'd.

For by the Property of the Circle FMN, it is manifest, 1. That the Chord FG is bisected in the Point P by the Diameter MN being * at right Angles to it; and therefore, (because $PH = PK$, by Construction) FH is $= GK$, and $GH = FK$; and consequently $GH \times HF = FK \times KG$. 2. $GH \times HF = \overline{HD}^2$, and $FK \times KG = \overline{KE}^2$, and so $HD = KE$. 3. If the Tangents HD, KE, be produced meeting one another in the Point Q, the Parts DQ, EQ, will be equal to one another; and so $DQ : EQ :: DH : EK$. Whence it appears, that the Line DE, joining the Points of Contact of the Tangents HD, KE, will be parallel to the Line HK, and the Plane SDE to the Plane CHK: Therefore the Line DE will be * the Directrix; and because the same cuts the Base in two Points, the Conick Section FAG will be * an Hyperbola. It is farther manifest, that this Hyperbola will pass through the given Point F, because this Point is both in the Conick Superficies and in the Plane HCK, being the Plane of the Hyperbola; and the Lines CH, CK, will be the Asymptotes of that Hyperbola, as being * the Common Sections of the touching Planes SDH, SEK, and the Plane of the Hyperbola.

* Hyp.

* Def. 9.

* Def. 10.

* Def. 15.

It may so happen that the Tangents DH, HK, be parallel to one another, and then it is manifest at Sight, that the Lines DE, HK, will be parallel to one another, because those Tangents are equal; and the rest of the Demonstration is the same as above.

PROPOSITION IX.

Theorem.

274. *If there be two Right Lines* MN, AB, *terminating in a Conick Section, or the opposite Sections meeting one another in the Point* P; *and if these Lines be parallel to two other Lines* SE, SD, *given in Position: I say the Rectangle* MP × PN *to the Rectangle* AP × PB, *will be in a given Ratio; that is, the Ratio of these two Rectangles will be always the same, let the Lines* MN, AB, *be any where drawn.* Fig. 141, 142.

Draw two Planes thro' the Parallels SE, MN, and SD, AB, then these Planes will form two Right Lines Enm, Dba, in the Plane of the Base, and the Sides of the Cone SMm, SNn, SAa, SBb, and their common Section will be the Line SPp, which meets the Plane of the Base in the Point p, wherein the two Right Lines Em, Da, cut one another; through which Point draw the Right Line HK in the Plane SMN, parallel to MN, and the Right Line FG in the Plane SAB, parallel to AB. This being done.

Because the Triangles SPM, SpH; SPN, SpK; SPA, SpF SPB, SpG, are similar, therefore $MP \times PN : Hp \times pK :: \overline{SP}^2 : \overline{Sp}^2 :: AP \times PB : Fp \times pG$. And therefore we have $MP \times PN : AP \times PB :: Hp \times pK : Fp \times pG$. And the Ratio of $Hp \times pK$, to $Fp \times pG$, is compounded of the Ratio's of $Hp \times pK$ to $mp \times pn$, and of $mp \times pn$, or, by the Property of the Circle, of $ap \times pb$ to $Fp \times pG$. But because the Triangles Hpm, SEm, and Kpn, SEn, are similar, therefore $Hp : mp :: SE : mE$. And $pK : pn :: SE : En$. And multiplying the Antecedents and Consequents of these two Ratio's, we have $Hp \times pK : mp \times pn :: \overline{SE}^2 : mE \times En$: In like manner, because the Triangles Fpa, SDa, and Gpb, SDb, are similar, therefore $ap \times pb : Fp \times pG :: aD \times Db : \overline{SD}^2$. Whence it is manifest, that the Ratio of $MP \times PN$ to $AP \times PB$, is compounded of the Ratio of $\overline{SE}^2$ to $mE \times En$, and of $aD \times Db$ to $\overline{SD}^2$. Which two last Ratio's, by the Property of the Circular Base of the Cone, always are the same wheresoever the Lines MN, AB are drawn, because the Points E, D do not vary. Therefore the Rectangle $MP \times PN$ to the Rectangle $AP \times PB$ is in a given Ratio. *W. W. D.*

COROLLARY.

FIG. 143, 144. 275. HENCE, in any Conick Section, or the opposite Sections, if there be any two parallel right Lines MN, OR, meeting a third right Line AB bounded by the Section, in the Points P, Q; I say, $MP \times PN : OQ \times QR :: AP \times PB : AQ \times QB$.

PROPOSITION X.

Theorem.

FIG. 145. 276. *IF the right Line* AB *be drawn through any Point* A *in a Parabola, or Hyperbola* MAN, *parallel to the Side* SD *of the Cone, which, if the Section be a Parabola, passes through the Point* D, *wherein the Directrix touches the Base; or, if an Hyperbola, through one of the Points wherein the same cuts the Base; and if the Line* MN *be drawn through any Point* P *in the said Line* AB, *parallel to the Line* SE *given in Position, and terminating in the Section, or opposite Sections, as also another Line* FG, *parallel to the Line* Da, *the common Section of the Plane* SAB, *and the Plane of the Base, and terminating in the Sides* Sa, SD; *I say, the Rectangle* MP × PN *to the Rectangle* FP × PG *is given, that is, it will be always the same in whatsoever Part of the Line* AB *the Point* P *be taken.*

If a Plane be drawn through the Parallels SE, MN, this will form the right Line Enm in the Plane of the Base; the Sides SMm, SNn, in the Conick Superficies; and the Line SPp in the Plane SDa, which Line meets the Base in the Point p, wherein the Lines Em, Da, intersect one another; and through this Point draw the Line HK in the Plane SMN, parallel to MN. This being laid down, the similar Triangles SPM, SpH; SPN, SPK; SPF, Spa; SPG, SPD, will give the following Proportions, $MP \times PN : Hp \times pK :: \overline{SP}^2 : \overline{Sp}^2 :: FP \times PG : ap \times pD$, or $mp \times pn$ (by the Property of the Circle.) And therefore we have $MP \times PN : FP \times PG :: Hp \times pK : mp \times pn$. But the Ratio of $Hp \times pK$ to $mp \times pn$ is compounded of the Ratio of Hp to pm, and of pK to pn, that is, (by the Similarity of the Triangles Hpm, SEm, and Kpn, SEn) of the Ratio of SE to Em, and of SE to En; and consequently $Hp \times pK : mp \times pn$, or $MP \times PN :: FP \times PG : \overline{SE}^2 : mE \times En$. Whence because the Position of the Point E is the same, let the Point P be taken at pleasure, and all the Rectangles $Em \times En$, by the Nature of the Circle are equal to one another; therefore $MP \times PN$ is to $FP \times PG$ in a given Ratio. *W.W.D.*

COROLLARY.

277. HENCE, in a Parabola or Hyperbola (MAN) if a Diameter (AB) be drawn through any Point A in the Parabola, or the Line AB parallel to one of the Asymptotes in the Hyperbola; and if the two Parallels MN, OR, are drawn from any two Points P, Q, in the Line AB, terminating in the Section or opposite Sections; then we shall have this Proportion, $MP \times PN : OQ \times QR :: AP : AQ$. FIG. 146.

For draw the Plane SAB forming the Sides of the Cone SD, Sa, such, that SD may pass through the Point wherein the Directrix touches the Base, when the Section is a Parabola, and through one Point wherein the Directrix cuts the Base, when the Section is an Hyperbola: Also thro' the Points P, Q, draw the right Lines FG, TV, in the Plane SDA parallel to Da; then it is evident, (by the last Proposition) that $MP \times PN : FP \times PG :: OQ \times QR : TQ \times QV$. And so $MP \times PN : OQ \times QR :: FP \times PG : TQ \times QV$. But FG, QV, the Parts of the Lines FG, TV, are equal to one another; because the Lines AB, SD, are parallel. And therefore $MP \times PN : OQ \times QR :: FP : TQ :: AP : AQ$ (by the Similarity of the Triangles APF, AQT. Whence, &c.

CHAP. II.

Of the Ellipsis only.

DEFINITIONS.

17.

Fig. 147. IF any indefinite right Line *SZ*, which is without the Plane of the Circle *VXT*, moves about the Circumference of that Circle always parallel to it ſelf, until it be returned to the ſame Place from which it went; then the Convex Superficies deſcrib'd by the Motion of the Line *SZ*, is call'd a *Cylindrick Superficies*, or the *Superficies of a Cylinder*.

18.

That Line *SZ* being in any different Poſition, is called always a *Side* of the Cylindrick Superficies.

19.

The Circle *VXT* is the *Baſe*.

20.

The indefinite right Line *CO*, drawn from *C* the Centre of the Baſe parallel to the Sides, is the *Axis* thereof.

21.

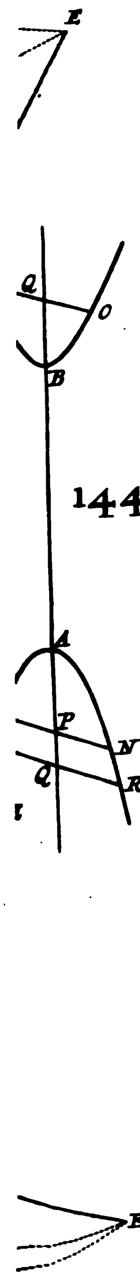
The indefinite Solid comprehended under the Baſe *VXT*, and the Cylindrick Superficies, is call'd a *Cylinder*.

22.

If a Cylinder be cut by any Plane not parallel to its Baſe, or the Sides thereof; then the Curve *AMBN* formed thereby in the Conick Superficies, is called a *Cylindrick Section*.

PRO

e 146



144

PROPOSITION XI.

Theorem.

278. *If any Cylinder be cut by a Plane* (v x y) *parallel to the Plane of the Base* (V X Y); *then the Section* v x y *will be a Circle, the Point* c *wherein that Plane meets the Axis, the Centre; and the Line* c x *equal to the Radius of the Base* C X, *the Radius thereof.* FIG. 147.

For if the Side of the Cylindrick Superficies $x X$ be drawn thro' any Point x in the Section $v x y$, this Side will be * parallel to the Axis Cc; therefore a Plane may be drawn through those two Lines forming the two right Lines CX, $c x$, in the parallel Planes $CVXY$, $cvxy$, parallel between themselves; which moreover will be equal to one another, as being included between the Parallels Cc, Xx. And because this is so always, let the Point x be taken any wheresoever in the Section $v x y$; therefore all the Lines $c x$, drawn from the Point c to the Points x in the Section $v x y$, are equal to the Radii CX of the Base; that is, the Section $v x y$ will be the Circumference of a Circle, the Point c, wherein the Plane $v x y$ meets the Axis of the Cylinder, the Centre, and the Line Cx equal to the Radius (CX) of the Base, the Radius thereof. *W. W. D.* * *Def.* 20.

PROPOSITION XII.

Theorem.

279. *Every Ellipsis may be considered as the Section of a Cylinder.* FIG. 148.

Draw the Diameter $a b$ in the Base of a Cone, wherein is produced any Ellipsis, meeting the Directrix DE at right Angles in the Point D; also draw the Sides $S a$, $S b$, of the Cone meeting the Plane of the Ellipsis in the Points A, B; and draw the right Lines AB, SD, in the parallel Planes AMB, SDE. This being done, assume (DF) a mean Proportional between aD, Db, draw AG, BH, parallel to SF, and describe a Circle upon the Plane of the Cone's Base, with the Line GH for a Diameter, as also a Cylindrick Superficies with that Circle as a Base; and the right Lines AG, BH, as Sides. This being premis'd.

If a right Line be drawn through any Point P in the Line AB, parallel to the Directrix DE, meeting the Conick Superficies in M, and the Cylindrick Superficies in O; then, I say, the Points M and O will coincide, or fall both in one Point.

For, if a Plane be drawn through that parallel Line, parallel to the Planes of the Base of the Cone and Cylinder; this Plane will form * * *Art.* 259.

the

the Circle KML in the Conick Superficies, whose Centre will be the common Section of that Plane, and the Axis of the Cone, together
* Art. 278. with the Circle* QMR in the Cylindrick Superficies, whose Centre will be the common Section of the aforesaid Plane, and the Axis of the
* Def. 6. Cylinder. But the Plane Sab does pass through * the Axis of the Cone, and the Plane $AGHB$ (coinciding with the Plane of the
* Def. 20. Triangle Sab) through * the Axis of the Cylinder; and consequently the Lines KL, QR, being the common Sections of those two Planes, and the Plane parallel to the Base passing through the Line POM, will be Diameters of the said two Circles; and the Line POM will be
* Hyp. perpendicular to those Diameters; because the same is * parallel to
* Hyp. DE, and DE is * perpendicular to ab and GH (both being in the
* Hyp. same * strait Line) and the Diameters KL, QR, (being both in the same strait Line) are parallel to ab and GH. Moreover, the Lines AB, SD, being the common Sections of two parallel Planes, and the Plane Sba, viz. the Plane SDE, and the Plane of the Ellipsis, will be equal to one another. This being well understood,

1. In the Cone, we have $\overline{PM}^2 = KP \times PL$, by the Nature of the Circle KML; and because the Triangles APK, SDa, and PBL, SDb, are similar, we shall get this Proportion, $AP : KP :: SD : aD$. And $PB : PL :: SD : Db$. Therefore $AP \times PB : KP \times PL$ or $\overline{PM}^2 :: \overline{SD}^2 : aD \times Db$.

2. In the Cylinder, we have $\overline{PO}^2 = QP \times PR$, by the Nature of the Circle QOR; and because the Triangles APQ, SDF, and PBR, SDF, are similar, we shall get the two following Proportions, viz. $AP : PQ :: SD : DF$; and $PB : PR :: SD : DF$; therefore $AP \times PB : QP \times QR$ or $\overline{PO}^2 :: \overline{SD}^2 : \overline{DF}^2$ or $aD \times Db$; and consequently $\overline{PM}^2 = \overline{PO}^2$, and $PM = PO$. Whence the Points M, O, coincide, or both fall in one. And because this happens always, let the Point P be taken any wheresoever in the Line AB, therefore the Plane of the Ellipsis meets both the Conick and Cylindrick Superficies in the same Points, and so any Ellipsis may be consider'd as a Cylindrick Section.

Advertisement.

Because a Cylinder is more simple than a Cone, in having all the Sides thereof parallel to one another, whereas the Sides of a Cone do all terminate in the Vertex; therefore we shall consider an Ellipsis in this Chapter, as being the Section of a Cylinder, and demonstrate the Properties of its Diameters from the Cylinder, which may be very easily done; and afterwards, in the next Chapter, we shall shew the extreme

extreme Facility in proving the same Properties of the Diameters of the Parabola and Hyperbola, by supposing Cones to have Elliptick Bases instead of Circular ones.

PROPOSITION XIII.

Theorem.

280. *ALL Diameters of an Ellipsis do cut one another in one Point only,* viz. *that Point wherein the Plane of the Ellipsis meets the Axis of the Cylinder, and are bisected in that Point.* FIG. 149.

And contrariwise, all Lines drawn through that Point, and terminating both ways in the Ellipsis, are bisected in that Point, and are Diameters of the Ellipsis.

Note, The aforesaid Point is call'd the *Centre* of the Ellipsis.

1. Let AB be any Diameter, and C the Point wherein the Plane of the Ellipsis meets the Axis of the Cylinder. Now if the Lines Aa, Bb, be drawn parallel to the Axis Cc, then it is manifest, * that the same will be Sides of the Cylindrick Superficies, and the two Planes FAa, GBb, passing through these two Lines, and the two Tangents AF, BG, (which by the Definition of Diameters must be parallel to one another) will be parallel between themselves, and touch the Cylindrick Superficies in the Sides Aa, Bb; whence those two Planes will form the Lines af, bg, in the Plane of the Base, parallel to one another, and touching the Base in the Points a, b, wherein the Sides Aa, Bb, meet it. But it is demonstrated, in the Elements of Geometry, that the Line ab joining the Points of Contact of two parallel Tangents (af, bg) to a Circle, passes through the Centre c; therefore the Plane $AabB$ will pass through Cc the Axis of the Cylinder; and the Line AB, which is the common Section of that Plane, and the Plane of the Ellipsis, will pass through the Point C, wherein the Axis meets the Plane of the Ellipsis. Moreover, because the Lines Aa, Bb, Cc, are parallel; it is manifest, that the Diameter AB of the Ellipsis is bisected in the Point C, because the Diameter ab of the Circle is bisected in the Centre (c) of the same: Which was to be demonstrated in the first place. * *Def.* 20.

2. If the Lines Aa, Bb, be drawn through A, B, the Extremities of any Line AB, (passing through the Centre C, wherein the Plane of the Ellipsis meets the Axis (Cc) of the Cylinder) parallel to that Axis; then it is manifest (by *Def.* 17.) that these Lines wil be Sides of the Cylinder, and the Plane $AabB$ will pass through the Axis Cc. Therefore the Line ab being the common Section of that Plane, and the Plane of the Base, will pass through c the Centre of the Base; and so since the same is bisected in c, the Line AB will be bisected also

in

in *C*. Moreover, because the Tangents *af*, *bg*, passing through the Extremities of the Diameter *ab*, are parallel to one another; therefore the touching Planes *faA*, *gbB*, will be parallel to one another, and form two parallel Lines *AF*, *BG*, in the Plane of the Ellipsis, which will touch the same in *A*, *B*, the Extremities of the Line *AB*, and so *AB* will be a Diameter of the Ellipsis. *Which was to be demonstrated in the second place.*

COROLLARY.

281. HENCE there can but one Diameter be drawn through a given Point (besides the Centre) in the Plane of an Ellipsis.

PROPOSITION XIV.

Theorem.

FIG. 149. 282. *Every Line* MPN *being an Ordinate on both Sides to any Diameter* AB, *is bisected by that Diameter in the Point* P.

And contrariwise, if any right Line MPN, *terminating in an Ellipsis, and not passing through the Centre* C, *be bisected by the Diameter* AB *in the Point* P; *then that Line will be an Ordinate on both Sides to that Diameter.*

Draw the Sides *Aa*, *Bb*, *Mm*, *Nn*, through the Points *A*, *B*, *M*, *N*, parallel to *Cc* the Axis of the Cylinder, and meeting the Plane of the Base in the Points *a*, *b*, *m*, *n*; then the Line *Pp*, being the common Section of the Planes *AabB*, *MmnN*, will be parallel to the Sides of the Cylinder, because all the Sides are parallel to one another. Moreover, the Plane *AabB* will pass through *Cc* the Axis of the Cylinder, because the Diameter *AB* passes through the Point *C*, wherein that Axis meets the Plane of the Ellipsis; and consequently does form a Line *ab* in the Plane of the Base, which passes through (*c*) the Centre, that is, a Diameter. This being laid down,

Because the Line *MPN*, (by Supposition) is an Ordinate both ways to the Diameter *AB*, the same will be parallel to the Tangents *AF*, *BG*, passing through the Extremities of that Diameter; and consequently the touching Planes *FAa*, *GBb*, will be parallel to the Plane *MmnN*. Therefore the Lines that those three Planes form in the Plane of the Base, *viz.* the two Tangents *af*, *bg*, and the Line *mn*, will be parallel to one another; and so the Line *mn* will be perpendicular to the Diameter *ab*, which consequently divides it into two equal Parts in the Point *p*. Therefore, because *Mm*, *Pp*, *Nn*, are parallel, the Line *MN* will be bisected likewise in the Point *P*.

Now for proving the Converse, draw two Tangents *AF*, *BG*, in

* Art. 267. the Plane of the Ellipsis, parallel * to *MN*; then if the Diameter

AB

AB be drawn through the Points of Contact; it is manifest (by the 13th and 14th Definitions) that the Line *MN* will be an Ordinate both ways to that Diameter; and consequently, by what has been already demonstrated, is bisected in *P* by the same. And because there can be drawn * but one Diameter through *P*, therefore if any Line *MN*, terminating in an Ellipsis, and not passing through the Centre *C*, be bisected by a Diameter *AB* in the Point *P*, then that Line *MN* will be an Ordinate both ways to *AB*. *Art. 281.

PROPOSITION XV.
Theorem.

283. *IN any Ellipsis, if there be two Diameters* AB, DE; *and if one of them, as* DE *be parallel to the Tangents* AF, BG, *passing thro' the Extremities of the other* AB; *then reciprocally, I say, the Diameter* AB *will be parallel to the Tangents passing through the Extremities of the Diameter* DE. FIG. 149.

The two Diameters *AB*, *DE*, are called *Conjugates* to one another.

Draw the Sides (*Aa*, *Bb*, *Dd*, *Ee*) of the Cylinder through the Points *A*, *B*, *D*, *E*, which meet the Plane of the Base in the Points *a*, *b*, *d*, *e*; then the Planes *AabB*, *DdeE*, shall pass through *Cc* the Axis of the Cylinder, because the Lines *AB*, *DE*, are Diameters of the Ellipsis; and consequently will form two Diameters *ab*, *de*, in the Plane of the Base. But the touching Plane *FAa*, being parallel to the Plane *DdeE*, will form a Tangent *af* in the Plane of the Base parallel to the Diameter *de*, which Diameter will be consequently perpendicular to the Diameter *ab*. Whence, if the Tangent *db* be drawn to the Circle, it will be parallel to *ab*, and the Plane *bdD* parallel to the Plane *AabB*; therefore the common Sections of those two Planes, and the Plane of the Ellipsis, *viz.* the Tangent *DH*, and the Diameter *AB*, are parallel to one another. We prove the same with regard to the Tangent passing thro' the other Extremity (*E*) of the Diameter *DE*. Therefore, &c.

COROLLARY I.

284. HENCE, in an Ellipsis, if *AB*, *DE*, be two Conjugate Diameters; then the two Planes passing through those Diameters, and the Axis *Cc* of the Cylinder, shall form two Diameters *ab*, *de*, in the Plane of the Base; which will be perpendicular to one another: And contrariwise.

COROLLARY II.

285. IT is manifest moreover by this Proposition, if the double Ordinate *MPN* be drawn to the Diameter *AB*, through any Point *P* in the same; that *MPN* will be parallel to the Diameter *DE*,

* Art. 275. which is a Conjugate to AB; and so we shall have * this Proportion, $MP \times PN$, or $\overline{PM}^2 : DC \times CE$, or $\overline{DC}^2 :: AP \times PB : AC \times CB$ or $\overline{AC}^2$. From whence we get $\overline{PM}^2 : AP \times PB :: \overline{DC}^2 : \overline{AC}^2 :: 4\overline{DC}^2$, or $\overline{DE}^2 : 4\overline{AC}^2$ or $\overline{AB}^2$; that is, the Square of any Ordinate (MP) to a Diameter (AB) to the Rectangle ($AP \times PB$) under the Parts of that Diameter, is as the Square of the Diameter DE, being a Conjugate thereto, to the Square of the Diameter AB.

PROPOSITION XVI.

Theorem.

Fig. 150. 286. *If through any Point* M *in the Ellipsis* AMB, *there be drawn a Tangent* FMG *meeting two other parallel Tangents* AF, BG, *in the Points* F, G; *I say,* FM : MG :: AF : BG.

Draw the Sides of the Cylinder Aa, Bb, Mm, through the Points of Contact A, B, M; and let three Planes FAa, GBb, FMm, or GMm pass through these Sides, and the Tangents AF, BG, FG; then it is plain, that the common Sections Ff, Gg, of the two first Planes, and the third, will be parallel between themselves, and to the Sides of the Cylinder; for since the two Planes FMm, FAa, pass through the Sides Mm, Aa, which are parallel to one another; their common Section Ff, will be parallel to those Sides; and by the same Reason Gg, the common Section of the two Planes GBb, GMm, will be parallel to the Sides Bb, Mm. Moreover, the Lines af, bg, which the parallel touching Planes FAa, GBb, form in the Plane of the Base, will be parallel Tangents to the Base; and fm, mg, the Parts of the third Tangent form'd in the Plane of the Base by the third touching Plane FMm, or GMm, shall be equal (by the Nature of the Circle) to the Tangents af, bg; viz. $fm = fa$, and $mg = gb$. This being premis'd:

Because the Lines Aa, Ff, Mm, Gg, Bb; as also AF, BG; and af, bg, are parallel, therefore $FM : MG :: fm$, or $fa : mg$, or $gb ::$ $FA : GB$. *W. W. D.*

Corollary I.

287. If a Diameter AB be drawn through A, B, the Points of Contact of two parallel Tangents AF, BG, meeting the Tangent FMG in T; and if the Ordinate MP be drawn to that Diameter; then it is manifest, that $AP : PB :: FM : MG :: AF : BG :: AT : BT$. And so $PB - AP : PB :: BT - AT$, or $AB : BT$.

Corol.

Corollary II.

288. HENCE arises the following Way of drawing a Tangent MT to touch an Ellipsis in the given Point M, by having a Diameter AB, and the Position of the Ordinates to the same given.

From B, one End of the Diameter AB, draw the right Line BM to the given Point M; then having drawn the Ordinate MP to the Diameter AB, and taken $PH = PA$ in the Diameter AB towards B, draw HK parallel to PM, meeting the Line BM in K, through which, and the other Extremity A draw AK. Finally, draw MT parallel to AK, and the same will be the Tangent requir'd.

For because MP, HK, and AK, MT, are parallel; therefore $BP : PH$, or $PA :: PM : MK :: BT : TA$.

Corollary III.

289. IN an Ellipsis, if there be two Tangents MT, NT, meeting one another in the Point T; I say, the Diameter AB passing through P the Middle of the Line MN joining the two Points of Contact, will pass likewise through the Point T. For PN, PM are each Ordinates to the Diameter AB; and consequently * the Tangents MT, NT will each meet that Diameter in one Point T, being such, that $PB - AP : PB :: AB : BT$; that is, in the same Point. *Art. 287.

Corollary IV.

290. IN an Ellipsis, if M, N, the Points of Contact of two Tangents MF, NL, be join'd by a right Line MN; and if some third Tangent FAL be parallel to MN; I say, FA, AL, the Parts of this last Tangent taken between the Point of Contact A, and the two first Tangents, shall be equal to one another. For draw the Diameter AB through the Point of Contact A. then it is manifest that the Line MN is an Ordinate both ways to that Diameter, because the same is parallel to the Tangent FL, passing through A the Extremity thereof; and so AB bisects MN in P, and consequently passes * thro' T, the Point of Concurrence of the Tangents MF, NL; or else will be parallel to them, if the Line MN be * a Diameter. And in both Cases it is evident, that FL will be bisected by the Diameter AB in the Point A; because MN is bisected by the same Diameter in the Point P. *Art. 289. *Art. 283.

CHAP. III.

Of the Parabola and Hyperbola only.

PROPOSITION XVII.

Theorem.

Fig. 151. 291. *IN the Parabola, every Ordinate* (MPN) *both ways to the Diameter* AB, *is bisected by that Diameter in the Point* P. *And contrariwise.*

Draw an Elliptick Plane through the Line *MN*, and this will form a Tangent *DE* in the touching Plane *SDE* (parallel to the Plane of the Parabola) parallel to *MN*. Farther, the Plane *SAF* drawn through *S* the Vertex of the Cone, and the Tangent *AF*, passing thro' *A* the Origin of the Diameter *AB*, will form the Tangent *af*, in the Elliptick Plane; and the Line *Da* joining the Points of Contact of the two Tangents *DE*, *af*, shall pass through the Point *P*; because the Diameter *AB* is parallel to the touching Side *SD*. This being laid down:

* Def. 14. Because the two Lines *AF*, *MN*, by Supposition, * are parollel to each other, therefore the Tangent *af*, being the common Section of the two Planes which pass through *AF*, *MN*, will be parallel to *MN*;
* Def. 13. and consequently parallel to *DE*. From whence * it appears, that the Line *Da*, which joins the Points of Contact of the two parallel Tangents *DE*, *af*, is a Diameter of the Ellipsis; and so the Line *MN*, which is parallel to those Tangents, and bounded by the Ellip-
* Art. 282. sis, will be bisected * in the Point *P*.

Now for proving the Converse: In the Plane of the Parabola, draw
* Art. 267. * the Tangent *AF* parallel to the Line *MN*, and the Diameter *AB*
* Def. 14. thro' the Point of Contact *A*; then the Line *MN* will be * an Ordinate both ways to the said Diameter, and will be bisected thereby in the Point *P*, as we have demonstrated. And because there is but one
* Art. 258. Diameter only, that can * pass thro' *P* the Middle of the Line *MN*; therefore, &c.

COROLLARY.

292. HENCE, if through any two Points *P*, *Q*, in the Diameter *AB*, there be drawn two Ordinates *MPN*, *OQR*, on both Sides thereof; it is manifest, that we shall have always * this

2 Propor-

Proportion, *viz.* $MP \times PN$ or $\overline{PM}^2 : OQ \times QR$ or $\overline{QO}^2 :: AP : AQ$. That is, the Squares of any two Ordinates PM, QO, to the Diameter AB, will be always to one another as the Parts AP, AQ, of that Diameter taken from the Origin A to the Points of Concurrence of the same Ordinates.

PROPOSITION XVIII.

Theorem.

293. *IF through any Point* M *in a Parabola, the Ordinate* M P *be drawn to any Diameter* A B, *as also the Tangent* M T *meeting that Diameter (produced beyond* A*) in the Point* T; *then, I say, the Parts* (A P, A T) *of the said Diameter, will be equal to one another.* FIG. 151.

The same Construction remaining as in the last Proposition, thro' the Vertex of the Cone S, and the Tangent MT, draw the touching Plane STM, which will form the Tangent MH in the Elliptick Plane, and meet the Diameter Da of the Ellipsis in the Point H thro' which the Line ST will pass; also draw the right Line TG parallel to SA. This being well understood, we shall have * this Proportion, $DH : Ha :: DP : Pa$, and (*alternando*) $DH : DP :: Ha : Pa$. But because AB, SD; and SA, TG, are parallel; therefore $DH : DP :: SH : ST :: Ha : Ga$. And so $Ha : Pa :: Ha : Ga$. Consequently $Pa = Ga$, and $AP = AT$. *W. W. D.* *Art. 287.

PROPOSITION XIX.

Theorem.

294. *IN the opposite Sections, every Diameter* A B *passes through* C, *the Point wherein the two Asymptotes cut each other, and is bisected by that Point: And contrariwise.* FIG. 152.

This Point is called the *Centre.*

Let HSb be one of the two common Sections of a Plane parallel to the Plane of the Hyperbola, and the opposite Superficies; and let FG be an Asymptote formed by the Concurrence of the Plane of the Hyperbola, and that Plane which touches the opposite Superficies in the Line HSb. Also through the parallel Tangents AF, BG, (passing through the Ends of the Diameter AB, and meeting the Asymptote FG in the Points FG) let there be drawn two parallel Elliptick Planes; then these Planes will form the parallel Tangents FH, Gbf, in the touching Plane passing thro' the Side HSb, and the parallel Tangents AF, af; in the touching Plane SAF.

This being premis'd, the Parallels FH, Gh, being included between the two Parallels FG, Hb, will be equal to one another; and

the

the similar Triangles SHF, Sbf, and SFA, Sfa, will give this Proportion, *viz.* $HF : bf :: SF : sf :: FA : fa$. And therefore $HF : FA :: bf : fa ::$ * $bG : GB$. And so since $HF = bG$, it follows, that $AF = BG$, and $AC = CB$, because the Triangles ACF, BCG, are similar; that is, the Asymptote FG passes through C the Middle of the Diameter AB. After the same way we prove, that the other Asymptote will pass through C. Whence it appears, that the Diameter AB passes thro' C the Intersection of the Asymptotes, and is bisected by the same.

* *Art.* 268.

Now let the Line AB, passing through the Intersection (C) of the Asymptotes, meet the opposite Sections in the Points A, B. And if the Tangent AF be drawn through the Point A, and the Tangent DG, be drawn * to the opposite Hyperbola parallel to AF; then it is evident (as has been demonstrated before) that the Line AD, which joins the Points of Contact of the Tangents AF, DG, being a Diameter, will pass through C the Intersection of the Asymptotes. And therefore the same will coincide with the Line AB, which passes * likewise through the same two Points A, C; that is, the Point D will coincide with the Point B. Whence the Line AB will be a Diameter, and consequently will be bisected in the Point C.

* *Art.* 267.
* *Hyp.*

Corollary.

295. HENCE it appears, that there can be drawn but one Diameter through a given Point within an Hyperbola; because there can but one Line only be drawn through that Point and the Centre.

PROPOSITION XX.

Theorem.

Fig. 153. 296. *IN the opposite Sections, every Ordinate* (MPN) *both ways to the Diameter* AB, *is bisected by that Diameter in the Point* P. *And contrariwise.*

Draw an Elliptick Plane through the Line MN, which will form two Tangents af, bg, in the touching Planes SAF, SBG; and the Line ab, joining the Points of Contact of those two Tangents, being the common Section of the Elliptick Plane, and the Plane SAB, will pass through the Point P. But because the two Lines AF, MN are parallel to one another (by Supposition) therefore the Line af, which is the common Section of the two Planes passing through AF, MN, will be parallel to MN. By the same Reason the Tangent bg being the common Section of the Elliptick Plane, and the touching Plane SBG, which pass through the two Parallels MN, BG, will be parallel to MN. Therefore the two Tangents af, bg will be parallel to one another: Whence it follows, that the Line ab * is a Diameter

* *Def.* 13.

meter of the Ellipsis; and so the Line MN* is bisected in the Point P. *Art. 282.

Now for proving the Converse, draw* two Tangents AF, BG, in the Plane of the Hyperbola's, parallel to the Line MN bounded by the Hyperbola; and draw the Diameter AB through their Points of Contact; then it is manifest (by *Def.* 14.) that MN will be an Ordinate both ways to that Diameter, and will be bisected by the same in P, as we have proved already; and because there is * only one Diameter that can pass through that Point, therefore if the Line MN, bounded by an Hyperbola, be bisected in P by the Diameter AB, the said Line MN will be an Ordinate both ways to that Diameter. *Art. 267. *Art. 295.

COROLLARY.

297. HENCE, if two Ordinates (MPN, OQR) be drawn (on both Sides) to the Diameter AB, we shall have always * *Art. 273.
$MP \times PN$ or $\overline{PM}^2 : OQ \times QR$ or $\overline{QO}^2 :: AP \times PB : AQ \times QB$.
That is, &c.

PROPOSITION XXI.

Theorem.

298. *IF through any Point* M *in an Hyperbola, there be drawn a Tangent* MFG, *meeting two other parallel Tangents* AF, BG, *in the Points* F, G; *I say,* MF : MG :: AF : BG. FIG. 154.

Draw two parallel Elliptick Planes thro' the Tangents AF, BG; these will form two parallel Tangents HF, bG, in the touching Plane SMG; and the Elliptick Plane passing through BG, will form the Tangent af, in the touching Plane SAF, which shall meet the Tangent bG in the Point f, wherein the Line FS meets that Elliptick Plane. This being laid down, the Tangents af, BG, shall be parallel to one another, because they are each parallel to the Tangent AF; and therefore * we shall have $BG : Gb :: af : fb :: AF : FH$; (because the Triangles Sbf, SHF, and Saf, SAF, are similar) whence $BG : AF :: Gb : FH :: MG : MF$ (because the Triangles MGb, MFH, are similar.) *W. W. D.* *Art. 268.

It is manifest, that the same Corollaries may be drawn from this Proposition as in the Ellipsis in the 287th, 288th, 289th, and 290th Articles, and so I shall here omit them.

PROPOSITION XXII.

Theorem.

299. *IF any right Line* FG, *terminating in the Asymptotes of an Hyperbola, touches the same in the Point* A; *I say, that Line will be bisected by the Point* A. FIG. 155.

Draw

Draw two Planes through S the Vertex of the Cone, and the two
* Def. 16. Asymptotes CF, CG, which shall touch * the Conick Superficies in the Sides SM, SN, wherein the Plane MSN parallel to the Hyperbolic Plane meets the same. Also draw an Elliptic Plane through the Right Line FG; which Plane will form two Tangents MF, NG, in the touching Planes, and in the Plane MSN, the Right Line MN parallel to FG, which joins the Points of Contact of the two Tangents ME, NG. This being done, it is manifest that the Line FG
* Art. 290. is * bisected in the Point A; because the same touches the Ellipsis as well as the Hyperbola in that Point.

COROLLARY I.

300. BEcause there is but one Line FG only, which passing through a Point A given within the Angle FCG, and being bounded by the Sides thereof, can be bisected by that Point A; therefore if a Right Line FG terminating in the Asymptotes of an Hyperbola, meets the Hyperbola in the Point A, dividing that Line FG into two equal Parts, the same shall touch the Hyperbola in that Point.

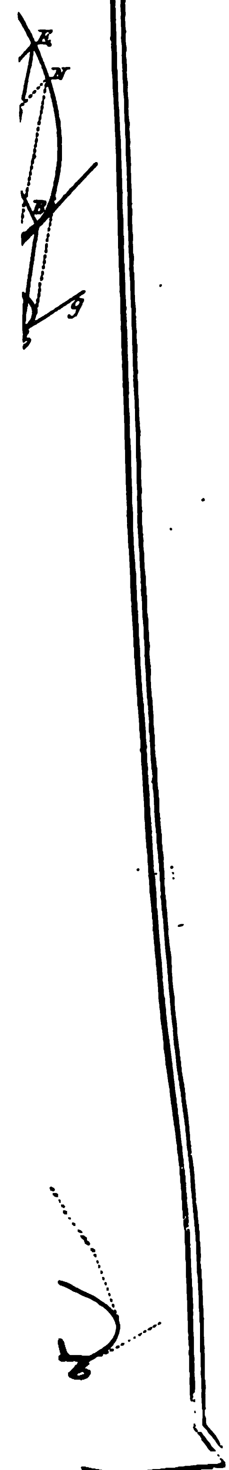
COROLLARY II.

301. HENCE if it be requir'd to draw a Tangent FAG from a Point A given in an Hyperbola, whose Asymptotes CF, CG, are given; then you need only draw the Line AD parallel to one of the Asymptotes CG, and terminating in the other: For if DF be taken equal to CD, and the Line FAG be drawn, the same will be the Tangent sought. For because the Triangles FCG, FDA,
* Hyp. are similar, the Line FG, shall be bisected in A; since CF is * in D.

COROLLARY III.

Fig. 156. 302. IF any two Points M, N, of an Hyperbola MAN be joyned by a Right Line meeting the Asymptotes in the Points H, K; then the two Parts of that Line MH, NK, contained between the Hyperbola and the Asymptotes, are equal between themselves. For if the Diameter CP be drawn through P the middle of MN, and the Line FG through the Point A, wherein that Diameter meets the Hyperbola, parallel to MN, and terminating in the Asymptotes;
* Art. 296. Then it is * manifest, that FG will be a Tangent in the Point A; and
* Art. 299. so will be * bisected in the same. Whence because the Triangles CAF, CPH, and CAG, CPK are similar, PH is $=PK$; and so $MH=NK$.

COROL-



COROLLARY IV.

303. IF through the Point A, given in an Hyperbola, there be drawn two right Lines AF, AG, terminating in the Asymptotes; and if from any other Point M of the same Hyperbola, or the opposite one, there be drawn two other right Lines MH, MK, likewise terminating in the Asymptotes, and parallel to the two former Lines AF, AG; I say $FA \times AG = HM \times MK$. FIG. 157.

For 1. When the two Points A, M, be in the same Hyperbola; then join them by a right Line meeting the Asymptotes in P and Q, and the similar Triangles PAF, PMH, and QMK, QAG, give the following Proportion, $AF : MH :: AP : MP$ * $:: MQ : AQ$ $:: MK : AG$. And so by multiplying the Means and Extremes, there arises $FA \times AG = HM \times MK$. *Art. 302.

2. When the Points A, M, are one in one Section, and the other in the opposite Section; draw the Diameter AB through the given Point A, and the Centre C, and draw the right Lines BD, BE, parallel to AF, AG, and terminating in the Asymptotes; then it is manifest, that the Triangles CAF, CBD, and CAG, CBE, will be similar and also equal, since * CA is $= CB$. Therefore $BD = AF$, and $BE = AG$; and so $DB \times BE = FA \times AG$. But (by the last Case) $KM \times MH = DB \times BE$. Whence also $FA \times AG = KM \times MH$. *Art. 294.

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I shall omit the other Properties of the Asymptotes and Conjugate Diameters, because these arise from those on a Plane, as is shewn in the third Book: My Design here having been only to shew the Usefulness of considering the Conick Sections in the Solid, and demonstrating immediately without any Calculus, those Properties of the Diameters, Tangents, and Asymptotes, from which all the other Properties may be taken: Which I have done (in my Opinion) after a very easy and new manner; in not having us'd Lines harmonically divided, as the modern Geometricians after M. *Paschal* and *Desargues* have: For this obliges them to have recourse to a great Number of Lemmata, whose Demonstrations alone (seem to me) to take up more room than this whole 6th Book.

The End of the Sixth Book.

BOOK VII.

Of Geometrick LOCI.

DEFINITIONS.

I.

FIG. 158, 159. IF there be two unknown and indeterminate right Lines, AP, PM, making any Angle (APM) with each other at pleasure; and if the Beginning of one of them, *viz.* AP, (which I call always x) be fixed in the Point A; and the said AP indefinitely extends it self along a right Line given in Position; and the other PM (which I call y) continually alters its Position, and is always parallel to it self. (That is, all the PM being parallel to one another.) Then if there be an Equation, wherein are both those unknown Quantities x and y mix'd with known ones, which expresses the Relation of every AP (x) to its Correspondent PM (y), the Curve passing thro' the Extremities of all the Values of y, that is, through all the Points M, is called in general a *Geometrick Locus*; and in particular, the *Locus* of that Equation.

FIG. 158. For Example: Let us suppose that the Equation $y = \frac{bx}{a}$ expresses always the Relation of the Line AP (x) to PM (y), which make any Angle APM at pleasure with one another: In the Line AP assume $AB = a$, and from B draw $BE = b$ parallel to PM, and on the same Side; then the indefinite Line AE is called in general a *Geometrick Locus*; and in particular, the *Locus* of the Equation $y = \frac{bx}{a}$. For if the right Line MP be drawn from any one of its Points M parallel to BE, the similar Triangles ABE, APM, will give always this Proportion, *viz.* $AB\ (a) : BE\ (b) :: AP\ (x) : PM\ (y) = \frac{bx}{a}$. And therefore the right Line AE is the Locus of all the Points M.

FIG. 159. Moreover, if $yy = aa - xx$ expresses the Relation of AP to PM, and the Angle APM be a right Angle; then the Circumference of a Circle, whose Radius is the right Line $AB = a$ taken in AP, is called in general a *Geometrick Locus*, and, in particular, the Locus of the

the Equation $yy = aa - xx$. For if the Perpendicular MP (y) be drawn from any Point M of the Circumference; then, by the Nature of the Circle, we shall have always $\overline{PM}^2 (yy) = DP \times PB (aa - xx)$ supposing BD the Diameter of the Circle. Therefore the Locus of all the Points M is the Circumference of a Circle.

SCHOLIUM.

304. If all the PMs be suppos'd to tend from one Side of the Line AB, as towards Q, and then they be suppos'd to tend from the other Side of the said Line, as towards G; then it must be observ'd, that their Values from Positives (which they are suppos'd to be when tending towards Q) will become Negative, and so we shall have $PM = -y$; moreover, if the Points P be suppos'd to fall from A towards B, and afterwards the contrary way, as from A towards D; then all the AP' on this Side A, will become Negative, and consequently we have $AP = -x$. And a Geometrick Locus must pass through the Extremities of all the Values (as well positive as negative) of one of the unknown Quantities y, which answer to the Values both positive and negative of the other unknown Quantity x. Therefore, if the right Line QAG be drawn parallel to PM, a Geometrick Locus may be found in the four Angles BAQ, BAG, GAD, DAQ, as in the second Example (*Fig.* 159.) or only in some of the Angles, as in the first Case (*Fig.* 158.) For in the second Example, suppose that AP be $= x$, and $PM = y$, the Point M being first taken on the Quadrant QB; then if the Point M be taken afterwards in the Quadrant GB, we shall have $AP = x$, and $PM = -y$; if M be taken on DG, we shall have $AP = -x$, and $PM = -y$; and finally, if M be taken on DQ, we shall have $AP = -x$, and $PM = y$; and in all these Cases (by the Nature of the Circle) there will come out the same Equation $yy = aa - xx$; because the Squares of $\pm y$, and $\pm x$, are the same in all Cases, *viz.* yy and xx. Moreover, in the first Example, if you make $AP = x$, and $PM = y$, in first taking the Point M (on the same Side as E) upon AE, in the Angle QAP; and then if the Point M be afterwards taken on EA (produced towards A) in the Angle GAD, we shall have $AP = -x$, and $PM = -y$; and since the Triangles ABE, APM, are similar, the following Proportion will be formed, *viz.* $AB, (a)\ BE\ (b) :: AP\ (-x) : PM\ (-y) = -\frac{bx}{a}$; and therefore $y = \frac{bx}{a}$. Which is the same Equation as was form'd by supposing the Point M to fall in the Angle BAQ. FIG. 158, 159.

Note, When we are hereafter to construct the Locus of a given Equation, we always suppose AP (x) and PM (y) positive,

 that

that is, all the Points M to fall in the same Angle BAQ. And that Part of the Locus contain'd in the Angle BAQ, we take for the Locus of the given Equation.

2.

The ancient Geometricians did call *plain Loci* such that are right Lines or Circles; and *solid Loci*, those that are Parabola's, Ellipses, or Hyperbola's. But the Moderns do distinguish Geometrick Loci into different Kinds or Degrees: For under the first Degree are comprehended all the Loci, wherein the unknown Quantities x, y, are found in Equations only of one Dimension; under the second, all those wherein those unknown Quantities have two Dimensions; under the third, all those wherein the unknown Quantities have three Dimensions, and so on. Where you must observe, that there must be no Rectangle, or Product of the unknown Quantities x and y in the Equations for the Loci of the first Kind or Degree; and in Equations for the second, those Quantities must form a Product as xy of no more than two Dimensions; and in Equations for the third, a Product xxy, or xyy of three Dimensions, &c.

3.

The Terms of the Equation of a Locus are said to be different, when either of the unknown Quantities x and y, or both of them together, are found therein of different Dimensions: So in the first Degree, if this Equation be propos'd $y - \frac{bx}{a} + c = o$, the Terms y, $-\frac{bx}{a}$, c, will be different. Moreover, in the second Degree, if you suppose $yy + \frac{2bxy}{a} - 2cy - \frac{fxx}{a} + gx + bx - bb + ll = o$, then the Terms yy, $\frac{2bxy}{a}$, $-2cy$, $-\frac{fxx}{a}$, $gx + bx$, $-bb + ll$, shall be every one of them different.

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I shall here only explain particularly the Loci of the first and second Degrees; but what I shall say thereof will give a great Insight into the Construction of more compound Loci in particular Cases that may occur; some Examples of which will be found hereafter: Therefore my Design in this Book, is to give a general Method for constructing the Loci of given Equations of the first and second Degrees; and to shew that the Loci of the first Degree are strait Lines only; and those of the second, either Parabolas, Ellipses, Circles, Hyperbolas, or the opposite Sections.

POSTU-

Postulate.

305. Grant that any given litteral Quantity, be it never ſo compounded, may be reduced to a ſimple Fraction in its leaſt Terms.

For Example: 1. Grant that one may take the ſimple Fraction $\frac{b}{a} = \frac{cc+ff}{af+fc} + \frac{aa}{gg}$, wherein the Letters a, c, f, g, denote given right Lines. 2. That one can find a right Line $s = \frac{age-bce}{bb+af}$, wherein the right Lines a, b, c, e, f, g, are given. 3. That one can find a Square $tt = ss - \frac{ccee-eebb}{bb+af}$, wherein the Lines a, b, c, e, f, b, s, are given; ſo that its Side t be $= \sqrt{ss - \frac{ccee-eebb}{bb+af}}$. We ſhall ſhew the manner of doing this, in the Beginning of the Eighth Book.

PROPOSITION I.
Problem.

306. *AN Equation of any Locus of the firſt Degree being given, to conſtruct the Locus.*

When the unknown Quantities x and y, have but one Dimenſion in a given Equation, and their Product xy is not in the ſame; then the Locus of that Equation will be always a ſtrait Line, and it may be reduced to ſome one of the four following Formula's.

1. $y = \frac{bx}{a}$, 2. $y = \frac{bx}{a} + c$, 3. $y = \frac{bx}{a} - c$, 4. $y = c - \frac{bx}{a}$,

in all which we ſuppoſe the unknown Quantity y to be freed from Fractions, and alſo that the Fraction multiplying the other unknown Quantity x be * reduced to this Expreſſion $\frac{b}{a}$, and all the known Terms to this, *viz.* c, * Art. 305.

The ſame Things being premis'd as in the firſt Definition, the Loci of the three laſt Forms may be conſtructed in the following manner; for the Locus of the firſt Form has been conſtructed in that Definition already.

Now to conſtruct the Locus of the ſecond Formula $y = \frac{bx}{a} + c$. In the Line AP aſſume $AB = a$; and draw the right Lines $BE = b$, $AD = c$, parallel to PM, and on the ſame Side; then if the indefinite right Line AE be drawn, as alſo the right Line DM parallel to AE. I ſay, this Line DM, contain'd in the Angle PAQ, (made FIG. 160.

by

by the Line AQ, drawn parallel to PM, and on the same Side) will be the Locus of the said Equation or Formula. For if through any one of its Points M, the Line MP be drawn parallel to AQ, meeting AE in F; then the similar Triangles ABE, APF, will give this Proportion, $AB\,(a) : BE\,(b) :: AP\,(x) : PF = \frac{bx}{a}$. And therefore $PM\,(y) = PF\left(\frac{bx}{a}\right) + FM\,(c)$.

FIG. 161. The Locus of the third Formula $y = \frac{bx}{a} - c$, may be constructed after the following manner: Assume $AB = a$, and draw the right Lines $BE = b$, $AD = c$, parallel to PM; *viz.* BE on the same Side as AQ, and AD on the contrary Side; also thro' the Points A, E, draw the Line AE of an indefinite length towards E, and thro' the Point D the Line DM parallel to AE, meeting the Line AP in G. I say, the indefinite right Line GM, contain'd in the Angle PAQ, will be the Locus sought. For we have always $PM\,(y) = PF\left(\frac{bx}{a}\right) - FM$ $(c.)$

FIG. 162. Lastly, To construct the Locus of the fourth Formula $y = c - \frac{bx}{a}$. In AP assume $AB = a$, and draw the right Lines $BE = b$, $AD = c$, parallel to PM; *viz.* BE on the contrary Side that AQ is, and AD on the same Side; then through the Points A, E, draw the Line AE of an indefinite Length towards E, and through the Point D the Line DM parallel to AE, meeting the Line AP in G. I say, the right Line DG, contain'd in the Angle PAQ, will be the Locus sought. For if the Line MP be drawn from any Point M thereof parallel to AQ, and meeting AE in F, we shall have always $PM\,(y) = FM$ $(c) - PF\left(\frac{bx}{a}\right)$.

If the unknown Quantity x be not multiplied by a Fraction; then the four foregoing Formula's will be changed into these here:

1. $y = x$, 2. $y = x + c$, 3. $y = x - c$, 4. $y = c - x$, all of which may be constructed after the same manner as before, only observing to take the right Line BE equal to AB, which before was taken of any Length at pleasure.

SCHOLIUM.

507. IT may happen that the Locus of an Equation may be a straight Line, although there be but one of the unknown Quantities x, y, contain'd therein; and from hence arises these two new Formula's, and $x = c$. 4

Now

Now to construct the first Formula $y=c$, the same things being always Fig. 163.
premised, as in Def. 1; through the fixed Point A draw the Right Line AD c parallel to PM and towards the same Parts, and then draw the indefinite Right Line DM parallel to AP: I say that Line DM, will be the Locus of the propos'd Equation. For if the Right Line MP be drawn from any Point M of the same, parallel to AD; then it is manifest that we shall have always $PM(y)=AD(c)$.

Again, to construct the second Formula $x=c$. Assume $AP=c$, and Fig. 164.
draw the indefinite Right Line PM making the Angle APM with AP, which is either given or taken at pleasure: Then I say, PM will be the Locus of all the Points M. For if the Right Line MQ be drawn from any Point M, thereof parallel to AP, and meeting the indefinite Line PQ (parallel to PM) in the Point Q; then it is manifest that we shall have always MQ or $AP(x)=c$, let $PM(y)$ be assum'd of any Length whatsoever.

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Here it will not be amiss to give the Learner an Idea of the Method I am going to use, in constructing the Loci of the second Degree. And this Method consists first in constructing a Parabola, and afterwards an Ellipsis and Hyperbola being such, that the Equation expressing the Nature thereof, with regard to the Diameters and Asymptotes, be the most compounded possible; and this furnishes general Equations or Formula's. Then I examine the particular Parts of these general Equations, that so an Equation being propos'd, I may know which of these general Formula's to compare the same with; this being known, and all the Terms of the Equation compar'd with those of the general Formula, I gather from thence the Construction of the Locus of that Equation, in observing certain Remarks serving for all the Formula's. But all this will be much better understood in the following Lemmata and Propositions.

The FUNDAMENTAL LEMMA *for the Construction of Loci, which are Parabola's.*

308. LET there be two unknown and indeterminate Right Lines Fig. 165,
$AP,(x)$, $PM(y)$ (as in Def. 1.); also let there be given 166.
Right Lines, as m, n, p, r, s. This being premised.

1. In the Line AP, assume $AB=m$; and draw the Right Fig. 165.
Lines $BE=n$, $AD=r$, parallel to PM, and towards the same Parts, also through the Point A draw the Right Line AE which I call e, and through the Point D the indefinite Right Line DG parallel to AE; and in DG assume $DC=s$ on the same Side as PM, and

with

with the Diameter CG, having its Ordinates parallel to PM, and Parameter $CH=p$, describe * a Parabola CM tending towards the same Parts as AP. Now I say the Portion of this Parabola contain'd in the Angle PAD, (form'd by the Line AP, and a Line AD drawn thro' the fixed Point A, parallel to PM, and towards the same Parts) is the Locus of the following Equation or Formula.

*Art. 161.

$$yy-\frac{2n}{m}xy+\frac{nn}{mm}xx-2ry+\frac{2nr}{m}x+rr=0.$$
$$-\frac{ep}{m}x+ps.$$

For through any Point (M) of that Portion of the Parabola, draw the Line MP, making the Angle APM (either given or taken at pleasure) with AP, and meeting the Parallels AE, DG, in the Points F, G; then the similar Triangles ABE, APF, will give these two Proportions; $AB\ (m):AE\ (e)::AP\ (x):AF$ or $DG=\frac{ex}{m}$. And $AB\ (m):BE\ (n)::AP\ (x):PF=\frac{nx}{m}$. And consequently GM or $PM-PF-FG=y-\frac{nx}{m}-r$, and CG, or $DG-DC=\frac{ex}{m}-s$. But the Parabola gives * $\overline{GM}^2=CG\times CH$, which Equation will be the same as the propos'd one, by putting for those Lines their analytick Values. Therefore, &c.

*Art. 19.

FIG. 166. 2. From the fix'd Point A, draw the indefinite Line AQ, parallel to PM, and towards the same Parts, in which assume $AB=m$, and draw $BE=n$, parallel to AP, and towards the same Parts as PM, and through the determinate Points A, E, draw the Line AE, which I call e; then if in AP you take $AD=r$ on the same Side as PM, and draw the indefinite Right Line DG parallel to AE, and if in the same you take $DC=s$, likewise on the same Side as PM, and afterwards with the Diameter CG, whose Ordinates are parallel to AP, and the Parameter $CH=p$, there be describ'd * a Parabola CM, tending the same way as AQ. I say the Portion of this Parabola contain'd in the Angle BAP, will be the Locus of this second Equation.

*Art. 161.

$$xx-\frac{2n}{m}yx+\frac{nn}{mm}yy-2rx+\frac{2nr}{m}y+rr=0.$$
$$-\frac{ep}{m}y+ps.$$

For if the Line MQ be drawn from any Point M thereof, parallel to AP, and meeting the Parallels AE, DG, in the Points F, G; then the similar Triangles ABE, AQF, will give these two Proportions

portions, $AB\ (m) : AE\ (e) :: AQ$, or $PM\ (y) : AF$ or $DG = \frac{ey}{m}$. And $AB\ (m) : BE\ (n) :: AQ\ (y) : QF = \frac{ny}{m}$. And consequently GM or $QM - QF - FG = x - \frac{ny}{m} - r$, and CG or $DG - DC = \frac{ey}{m} - s$. But the Parabola gives $\overline{GM}^2 = CG \times CH$, which Equation will be the same as the proposed one, by putting for those Lines their Analytick Values; therefore, &c.

COROLLARY.

309. IT is manifest, (1.) That in the former of the two Equations, or Formula's, the Square yy is found without a Fraction, and in the 2d the Square xx. 2. In both the Formulas the two Squares xx and yy are found with the same Signs; so that the Square $\left(\frac{nn}{mm}\right)$ of half the Fraction $\frac{2n}{m}$, which multiplies the Plane xy, multiplies xx and yy. And so if the Plane xy be in neither of those two Equations, then the Square $\frac{nnxx}{mm}$, or $\frac{nnyy}{mm}$ will also be not in them, because the given Fraction $\frac{2n}{m}$ will then be equal to nothing.

PROPOSITION II.

Problem.

310. *TO construct the Locus of a given Equation, wherein if the Plane* xy *be not, there will likewise be but one of the Squares* xx *and* yy; *or else if the Plane* xy *be found in the same, the two Squares* xx *and* yy *are also therein both with the same Signs; so that the Square of half the Fraction multiply'd by* xy, *be equal to that which multiplies the Square of one of the unknown Quantities: Supposing always one of the Squares* xx *or* yy *in the given Equation to be free from Fractions.*

Compare each Term of the given Equation with that Term answering to it in the first Formula of the foregoing Lemma, if the Square yy happens to be without a Fraction; or with that Term answering to it in the second Formula, when the Square xx is without a Fraction. Then by comparing these Terms, get the Values of the Quantities m, n, p, r, s; by means of which, if a Parabola be describ'd according as is directed in the Lemma (using the two following Observations) the same will be the Locus requir'd.

REMARK I.

311. 1. THE Line AB (m) may be taken of any positive Magnitude at pleasure. 2. The Lines AB (m), BE (n) being given, the Line AE (e) is given also, because the Angle ABE is given. 3. When $n = o$, the Line AE falls in AB, that is, in AP in the Construction of the first Formula, and in AQ in that of the second; then we shall have AB (m) $-$ AE (e); because the Points B, E, will coincide. 4. When the Value of one of the Quantities n, r, s, is negative, then the Line that the same expresses must be taken or drawn on the contrary Side AP with regard to PM; whereas when it is positive, it must be drawn on the same Side, as it is in the Lemma.

REMARK II.

312. IF the Value of the Parameter CH (p) happens to be negative, then the Parabola must be drawn the contrary way to that in the Lemma; that is, on the opposite Parts to which the indeterminate Line AP, in the Construction of the first Formula, and the indeterminate Line AQ in the second, tends. All this will be manifest by the following Examples.

EXAMPLE I.

313. LET there be a given Equation $yy - 2ay - bx + cc = o$; it is requir'd to construct the Locus of the same.

Because the Square yy is here without a Fraction, therefore I chuse * the first Formula of the Lemma; and comparing each Term thereof answering it in the propos'd Equation, (1.) I have $\frac{2n}{m} = o$, because the Plane xy being not in the proposed Equation, that Plane may be esteem'd as multiply'd by o; from whence I get $n = o$, and consequently * $m = e$: Therefore, if all the Terms in the Equation affected with $\frac{n}{m}$ be struck out, and m be put for e the Value thereof, there will arise $yy - 2ry - px + rr + ps = o$. 2. By comparing the correspondent Terms $-2ry$ and $-2ay$, as also $-px$ and $-bx$, I get $r = a$, and $p = b$. 3. By comparing the Terms not affected with the unknown Quautities x and y, I have $rr + ps = cc$; and so if a and b be put for their Values r and p, there arises $s = \frac{cc - aa}{b}$, which is a negative Value when a exceeds c, as here suppos'd. There is no need of comparing the first Terms yy, because they are exactly the same. Now the Values of n, r, p, s, being thus determin'd, I construct the Locus by using the Construction of the Equation *, and observing the first * Remark, after the following manner. Because

* Art. 308. n. 1.

* Art. 311.

* Art. 308. n. 2.

Because $BE(n) = o$, therefore the Points B, E, do coincide, and the Line AE falls * in AP; whence thro' the fix'd Point A, I draw first the Line $AD(r) = a$ parallel to PM, and on the same Side AB, as PM, because the Value thereof is positive: Then I draw DG parallel to AP, and in the same assume $DC = \frac{aa-cc}{b} = -s$ on the contrary Side to PM; because s is $= \frac{cc-aa}{b}$, which is a negative Value. And lastly, if a Parabola * be described with the Diameter CG, (whose Ordinates are parallel to PM) and Parameter $CH(p) = b$: I say the two Portions OMM, RMS, contain'd in the Angle PAO, (made by AP, and the Line AO drawn parallel to PM, and towards the same Parts) will be the Locus of the given Equation. *Art. 311. Fig. 167. *Art. 161.

For if the Line MP be drawn from any Point M of these two Portions, making with AP the Angle APM either given or taken at pleasure, and meeting DG in the Point G; then we have $GM = y - a$, or $GM = a - y$, according as the Point M be taken above or below the Diameter CG; and CG or $DG + CD = x + \frac{aa-cc}{b}$; and therefore, by * the Property of the Parabola $\overline{GM}^2\ (yy - 2ay + aa) = CG \times CH\ (bx + aa - cc)$ that is, $yy - 2ay - bx + cc = o$, which is the Equation given. Therefore, &c. *Art. 19.

Scholium.

314. If AO be produced on the other Side of A towards X, then observe,

1. That the indefinite Portion SM of the Parabola, contain'd in the Angle SAX, will be the Locus of all the negative Values of the unknown Quantity y, answering to the positive Values of the other unknown Quantity x in the given Equation. For if AP be assum'd greater than AS, and PM be drawn parallel to AX, and towards the same Parts, meeting the Portion SM in M; then we shall have * $PM = -y$, and so the right Line GM or $GP + PM = a - y$, and by the Property of the Parabola we shall get the given Equation again, as above. *Art. 304.

2. The Portion RCO of that Parabola, which falls in the Angle TAO vertically opposite to the Angle SAX, will be the Locus of all the positive Values of the unknown Quantity y in the given Equation, answering to the negative Values of the other unknown Quantity x; for if you make * $AP = -x$, we shall get the given Equation again. *Art. 304.

3. If a Portion of the said Parabola falls in the Angle TAX vertically opposite to the Angle PAO, the same will be the Locus of all

the negative Values of the unknown Quantity y, anſwering to the negative Values of the other unknown Quantity x. So that that Parabola is the compleat Locus of all the poſitive and negative Values of y, anſwering to all the poſitive and negative Values of x, in the given Equation $yy - 2ay - bx + cc = o$.

Hence it appears, that in this Example there are two poſitive Values (PM, PM,) of the unknown Quantity y, which anſwer to the ſame poſitive Value (AP) of the other unknown Quantity x, when the Line AP is leſs than AS; that there is one poſitive Value PM, and a negative one $-PM$, when AP exceeds AS; that there is but one poſitive Value (SV) of y, the other being $= o$, when $AP = AS$; that there are two poſitive Values (PM, PM) of y, anſwering to the ſame negative Value ($-AP$) of x, when AP is leſſer than AT; that thoſe two Values will become each equal to the Tangent TC, when $AP = AT$; and finally, if AP ($-x$) be taken greater than AT, then PM apply'd to AP, will not meet the Parabola at all, and ſo in this Caſe there can be had neither a poſitive or negative Value of y, that can anſwer to the negative Value ($-AP$) of x; that is, the Values of y will then become imaginary.

All this muſt be underſtood after the ſame manner in all the following Examples, as well in the other Conick Sections, as the Parabola: So that the Conick Section deſcrib'd, will not be only the Locus of all the poſitive Values of y, with regard to the poſitive Values of x; but the ſame will be likewiſe the Locus of all the poſitive and negative Values of y, with regard to all the poſitive and negative Values of x.

EXAMPLE II.

315. LET $yy + \frac{2b}{a}xy + \frac{bb}{aa}xx + 2cy - bx + cc = o$, be an Equation, whoſe Locus is requir'd to be conſtructed.

Becauſe the Square yy is here found without a Fraction, therefore
* Art. 308. n. 1. I chuſe, (as before) the firſt Formula * of the Lemma; then by comparing the Terms thereof with thoſe anſwering to them in the
* Art. 311. propos'd Equation, I have 1. $\frac{2n}{m} = -\frac{2b}{a}$; from whence making * $m = a$, there ariſes $n = -b$. 2. $\frac{nn}{mm} = \frac{bb}{aa}$; from whence there comes out, as above, $n = -b$. 3. $r = -c$. 4. $\frac{2nr - ep}{m} = -b$; and therefore $p = \frac{ab + 2bc}{e}$, by putting a, $-b$, $-c$, for m, n, r. 5. $rr + ps = cc$, and ſo $s = o$, by ſubſtituting cc for rr. Now the Values of m, n, r, p, s, being thus determin'd, I conſtruct the Locus of the Equation
* 308. (by uſing the Conſtruction of the firſt * Formula) after the following

In the Line AP assume AB $(m) = a$, and draw the right Lines $BE = b = -n$, $AD = c = -r$ parallel to PM, and on the contrary Side, because $n = -b$, and $r = -c$, which are negative Values. Then through the determinate Points A, E, draw the Line AE (e) which is given, and through the Point D, the Line DG parallel to AE. This being done, since DC (s) is $= o$, the Point C falls on D; therefore if a Parabola be describ'd * with the Diameter DG, whose Ordinates are parallel to PM, and the Parameter DH $(p) = \frac{+2bc}{e}$; I say, the Portion OM thereof contain'd in the Angle PAH, wherein all the Points M are suppos'd to fall, will be the Locus of the Equation given. FIG. 168. *Art. 161.

For if the Line MP be drawn through any Point M thereof, making the Angle APM either given or taken at pleasure, and meeting the Parallels AE, DG, in the Points F, G; then the similar Triangles ABE, APF, will give these two Proportions, AB $(a) : AE$ $(e) :: AP$ $(x) : AF$ or $DG = \frac{ex}{a}$. And AB $(a) : BE$ $(b) :: AP$ $(x) : PF = \frac{bx}{a}$. And consequently GM or $PM + PF + FG = y + \frac{bx}{a} + c$. But by the Property of the * Parabola $\overline{GM}^2 = GD \times DH$; that is, by putting the Analytick Values, $yy + \frac{2b}{a}xy + \frac{bb}{aa}xx + 2cy - 2bx + cc = o$. Therefore, &c. *Art. 19.

S C H O L I U M I.

316. IF the Line AP should not cut the Parabola, but touch or fall quite without it; then not one of the sought Points M would fall in the Angle PAH, as we have suppos'd in the Construction; and so there could be no positive Value of x, that would answer to the positive Value of y. FIG. 168.

This Observation is general for all Examples of the like Nature, not only in the Paarabola, but also in the other Sections.

S C H O L I U M II.

317. HERE it is necessary to take notice, that if AB (m) had been assum'd of any other Length besides a; then the Values of BE (n), and AE (e) would indeed vary: But the Ratio's of $\frac{n}{m}$, $\frac{e}{m}$, will remain the same always; because in the Triangle ABE, the Angle ABE is given, as also the Ratio of the Sides AB, BE, *viz.* $\frac{n}{m} = \frac{b}{a}$ in this Example. Now, because there

are

are only those two Ratio's of $\frac{n}{m}$, $\frac{e}{m}$, that can be found in the Values of p, r, s; therefore these Values do remain the same always, let the Line AB (m) be taken of any positive Magnitude at pleasure: So that m was taken equal to a only, for rendring the Construction more simple; which must be always observ'd hereafter.

EXAMPLE III.

318. IT is requir'd to find the Locus of this given Equation, $xx + \frac{2b}{a}yx + \frac{bb}{aa}yy - 2cx + by - \frac{2bc}{a}y = 0$.

Because the Square xx is here free from Fractions, I chuse the second Formula * of the Lemma; then in comparing the correspondent Terms, I have 1. $\frac{2n}{m} = -\frac{2b}{a}$; from whence making $m = a$, I get $n = -b$. 2. $\frac{nn}{mm} = \frac{bb}{aa}$; and so (since $m = a$) we have $n = -b$, as before. 3. $r = c$. 4. $\frac{2nr - ep}{m} = b - \frac{2bc}{a}$; from whence $p = -\frac{ab}{e}$; by substituting $a, -b, c$, for their Values m, n, r. 5. $rr + ps = 0$, because there is no Term in the given Equation entirely known, which can be compar'd with the Term $rr + ps$ of the Formula; and so we have $s = -\frac{rr}{p} = \frac{ecc}{ab}$, by substituting c and $-\frac{ab}{e}$, for r and p. Now these Values being thus determin'd, I construct the Locus requir'd after the following manner, by using the Construction of the second Formula * of the Lemma, and exactly observing the 311th and 312th Articles.

*Art. 308. n. 2.

*Art. 308. n. 2.

FIG. 169. Through the fixed Point A, draw the indefinite right Line AQ parallel to PM, and on the same assume AB (m) $= a$; and from the Point B draw $BE = b = -n$ parallel to AP, on the contrary Side to PM, because the Value of n is negative; moreover, through the determinate Points A, E, draw the Line AE, (e) which is given. This being done, in AP take AD (r) $= c$ in from A towards PM, and draw the indefinite right Line DG parallel to AE, in which assume

*Art. 161. DC (s) $= \frac{ecc}{ab}$ on the same Side AP as PM. Then describe * a Parabola with the Diameter CG, whose Ordinates are parallel to AP, and

*Art. 312. parameter the Line $CH = \frac{ab}{e} = -p$, tending * the contrary way to what which AQ tends, because $p = -\frac{ab}{e}$ which is a negative Va-

lue

lue. I say the Portion OMR of that Parabola, contained in the Angle PAB, will be the Locus sought.

For if the Line MQ, be drawn through any Point M thereof, parallel to AP, and meeting the Parallels AE, DG, in the Points F, G; then the similar Triangles ABE, AQF, will give these two Proportions, $AB\ (a) : AE\ (e) :: AQ$ or $PM\ (y) : AF$ or $DG = \frac{ey}{a}$. And $AB\ (a) : BE\ (b) :: AQ\ (y) : QF = \frac{by}{a}$. And consequently $GM\ (QM + QF - FG) = x + \frac{by}{a} - c$, or $GM\ (FG - FQ - QM) = c - \frac{by}{a} - x$, according as, on which Side of the Diameter CD the Point M falls; and CG or $CD - DG = \frac{ecc}{ab} - \frac{ey}{a}$. But by the Property * of the Parabola, $\overline{GM}^2 = CG \times CH$; that is, by substituting the Analytick Values of those Lines, $xx + \frac{2b}{a}yx + \frac{bb}{aa}yy - 2cx + by - \frac{2bc}{a}y = o$, which is the given Equation. Therefore, &c. * Art. 19.

SCHOLIUM.

319. IF it should happen, in comparing the Terms of the given Equation with those of the Formula, that p is $= o$; then it is manifest, that the Construction of the Parabola, which ought to be the Locus of the Equation, would be impossible; and that Equation may always be brought lower, so that the Locus thereof will be a strait Line; as will appear by the Formula's * of the Lemma. For Example; if all the Terms affected with p in the first Formula be struck out, there will arise $yy - \frac{2n}{m}xy + \frac{nn}{mm}xx - 2ry + \frac{2nr}{m}x + rr = o$, the square Root of which being extracted, will be $y - \frac{nx}{m} - r = o$, or $y = \frac{nx}{m} + r$, whose Locus is a strait Line, and may be constructed by *Art.* 306. The same thing will happen also in the second Formula of *Art.* 308. * Art. 308.

EXAMPLE IV.

320. LET there be an Equation $xx - ay = o$, it is requir'd to find the Locus of the same.

Because the Square xx is free from Fractions, I chuse the second Formula * of the Lemma; then by comparing the correspondent Terms, * Art. 308. n. 2.

Terms, I have 1. $\frac{2n}{m} = o$, because xy is not in that Equation; whence

* *Art.* 311. I get $n = o$, and consequently m * $= e$. 2. $\frac{nn}{mm} = o$, because the Square yy is not in the Equation; from whence I get again $n = o$. 3. $r = o$, because the unknown Quantity x is not found, in the proposed Equation, of one Dimension: Therefore striking out all the Terms in the Formula wherein are $\frac{n}{m}$ and r, and substituting m for e; then there arises $xx - py + ps = o$, whose Terms remain to be compared with those answering to them in the proposed Equation. 4. By comparing the Terms $-py$ and $-ay$, I get $p = a$. 5. Because there is no Term in the proposed Equation entirely known to compare with the Term ps; therefore $ps = o$, and so s is $= o$. Now the Values of n, r, p, s, being thus determin'd, the Locus required may be constructed in the following Manner, regard being had to the Construction of the second Formula of *Art.* 308, and *Art.* 311.

FIG. 170. Because $BE\ (n) = o$, the Line AE falls * in AQ drawn parallel

* *Art.* 311. to PM towards the same Parts; as also DG, because $AD\ (r) = o$. And because $CD\ (s) = o$ the Point C falls on D, and D on A. Then

* *Art.* 161. if a Parabola be described * with the Diameter AQ, whose Ordinates are Right Lines MQ parallel to AP, and Parameter $AH\ (p) = a$: I say the indefinite Portion (AM) thereof, contain'd in the Angle PAQ, is the Locus sought.

For if through any Point M thereof, you draw the Right Lines

* *Art.* 19. MP, MQ, parallel to AQ and AP; then by * the Property of the Parabola, we shall have $\overline{QM}^2\ (xx) = AQ \times AH\ (ay)$; and therefore $xx - ay = o$, which is the Equation proposed. *W. W. D.*

The Demonstration of the PROBLEM.

* *Art.* 308. 321. IF instead of m, n, r, s, p, in the general Formula, * you substitute the Values found by comparing the Terms thereof with the Terms of a proposed Equation, be it what it will, provided the same to have the Conditions denoted in the Problem; then it is manifest that that general Formula will be changed into the proposed Equation: And therefore if those Values be taken also in the Con-

* *Art.* 308. struction of the * Lemma, then the Locus of the general Formula will be changed into that of the proposed Equation. And this is what hath been taught in the Problem, as appears fully in the foregoing Examples. Therefore, &c.

The

The FUNDAMENTAL LEMMA, *for the Construction of Loci, which are Ellipses or Circles.*

322. LET $AP\,(x)$, $PM\,(y)$, be two unknown and indeterminite Right Lines (as in the first Definition) and let m, n, p, r, s, t, be given Lines. This being premised. FIG. 171.

In the Line AP, assume $AB = m$, and draw the Right Lines $BE = n$, $AD = r$, parallel to PM, and on the same Side AP as PM, also thro' the Point A draw the Right Line $AE\,(e)$ which is given, and thro' the Point D, the indefinite Right Line DG parallel to AE, in which assume $DC = s$ towards PM; and the Parts CK, CL, on both Sides C, each equal to t. Then if an Ellipsis be describ'd * with the Diameter $LK\,(2t)$ whose Ordinates are parallel to PM, and having $HK = p$ for the Parameter; I say the Portion OMR thereof contain'd in the Angle PAD, made by the Line AP and the Line AD (drawn thro' the fixed Point A parallel to PM, and the same way) will be the Locus of the following general Formula. *Art. 161.

$$yy - \frac{2n}{m}xy + \frac{nn}{mm}xx - 2ry + \frac{2nr}{m}x + rr = 0.$$
$$\qquad\qquad + \frac{eep}{2mmt} \qquad\qquad - \frac{2eps}{2mt} - \frac{ptt}{2t}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad + \frac{pss}{2t}$$

For if from any Point M of that Portion of the Ellipsis, you draw the Line MP, making (with AP) the Angle APM either given or taken at pleasure, and meeting the Parallels AE, DG, in the Points F, G; then by means of the similar Triangles ABE, APF, we shall have AF or $DG = \frac{ex}{m}$, and $PF = \frac{nx}{m}$. And therefore $GM = y - \frac{nx}{m} - r$, and $CG = \frac{ex}{m} - s$. And by the Property * of the Ellipsis, $KL\,(2t) : KH\,(p) :: LG \times GK$ or $\overline{CK}^2 - \overline{CG}^2 \left(tt - ss + \frac{2esx}{m} - \frac{eexx}{mm}\right) : \overline{GM}^2 \left(yy - \frac{2n}{m}xy - 2ry + \frac{nn}{mm}xx + \frac{2nr}{m}x + rr\right) = \frac{ptt - pss}{2t} + \frac{2epsx}{2mt} - \frac{eepxx}{2mmt}$. Therefore, &c. * Art. 55 and 41.

If it happens that the Diameter $KL\,(2t)$ and its Parameter $KH\,(p)$ be equal to one another, we shall have always $\overline{GM}^2 = LG \times CK$; from whence if the Angle CGM be a right one, it is manifest (by the Elements) that the Ellipsis will be then changed into a Circle, whose Diameter is the Line KL.

 COROL-

COROLLARY.

323. IT is evident that the two Squares yy and xx have always the same Signs in that Formula; and when the Plane xy happens to be therein, then the Square $\left(\frac{nn}{mm}\right)$ of half the Fraction $\frac{2n}{m}$ multiplying that Plane, must be lesser then the Fraction $\frac{nn}{mm}+\frac{eep}{2mmt}$ multiplying the Square xx.

PROPOSITION III.

Problem.

324. *TO construct the Locus of a given Equation, wherein are the two Squares* yy, *and* xx, *having the same Signs, without the Plane* xy, *or with it, if the Square of half the Fraction multiplying the same be less than the Fraction multiplying the Square* xx. *Supposing the Square* yy *to be always free from Fractions.*

Compare the Terms of the given Equation with those Terms
*Art. 322. answering to them in the general Formula * of the aforesaid Lemma; from whence get the Values of the Quantities m, n, p, r, s, t, by means of which describe an Ellipsis according to the Directions of the Lemma, (observing exactly the 311th Article) and the same will be the Locus sought.

EXAMPLE I.

325. IT is requir'd to find the Locus of this Equation, $yy + xy + \frac{1}{2}xx - 2ay + bx + cc = 0$, in which the Square of $\frac{1}{2}$ the Fraction $\frac{1}{1}$ or 1, which multiply's xy, is less then the Fraction $\frac{1}{2}$ which multiplies xx.

Now by comparing each Term of the general Formula of the Lem-
*Art. 322. ma * with that answering to it in the Equation, we shall have 1. $\frac{2n}{m} = -1$, for since the Plane xy is not multiply'd by any litteral Fraction, the same may be consider'd as being multiply'd by unity or 1: And consequently if you make $m = a$, we shall have $n = -\frac{1}{2}a$. 2. $\frac{nn}{mm}+\frac{eep}{2mmt}=\frac{1}{2}$; from whence we get $\frac{p}{t}=\frac{mm-2nn}{ee}=\frac{aa}{2ee}$, by substituting $a, -\frac{1}{2}a$, for m, n; and consequently $p=\frac{aat}{2ee}$. 3. $r = a$. 4. $\frac{2nr}{m} - \frac{eeps}{2mt} = b$; whence putting $a, -\frac{1}{2}a, a, \frac{aa}{2ee}$, for their Values $m, n, r, \frac{p}{t}$, and there arises $s = \frac{-2ae-2eb}{a}$. 5. $rr - \frac{ptt}{2t}+\frac{pss}{2t} = cc$: And therefore $tt =$

$ss +$

$ss + \frac{2arr}{p} - \frac{2bce}{p} = ss + 4ee - \frac{4ccee}{aa}$, by putting $\frac{aa}{2ee}$ for $\frac{p}{t}$, r. Now the Values of m, n, r, s, t p, being thus found, the Ellipsis sought may be describ'd after the following Manner, using the Construction of the Lemma * and the 311th Article. *Art. 322.

In the Line AP assume AB $(m) = a$, and draw the right Line AD $(r) = a$ parallel to PM, and on the same Side, as also the right Line $BE = \frac{1}{2}a = -n$ on the contrary Side, because $n = -\frac{1}{2}a$ is a negative Value; moreover, through the Point A draw the right Line AE (e) which is given, and through the Point D the right Line DG parallel to AE; in which assume $DC = \frac{2ae+2be}{a} = -s$ on the contrary Side to PM, as also CK, CL, on both Sides C, each equal to $t = \sqrt{ss + 4ee - \frac{4ccee}{aa}}$. Then if an Ellipsis be * describ'd with the Diameter LK, whose Ordinates are parallel to PM, and Parameter the Line KH $(p) = \frac{aat}{2ee}$. I say, the Portion (OMR) thereof, contain'd in the Angle PAD, will be the Locus of the given Equation. FIG. 172. *Art. 161.

For if thro' any Point M thereof, you draw the Line MP, making an Angle (APM) with AP, either given or taken at pleasure, and meeting the Parallels AE, DG, in the Points F, G; then the similar Triangles ABE, APF, will give these Proportions, AB $(a) : AE$ $(e) :: AP$ $(x) : AF$ or $DG = \frac{ex}{a}$. And AB $(a) : BE$ $(\frac{1}{2}a) :: AP$ $(x) : PF = \frac{1}{2}x$. Therefore we have $GM = y + \frac{1}{2}x - a$; and CG or $DG + DC = \frac{ex}{a} - s$, because $DC = -s$. But by the Property * of the Ellipsis KL $(2t) : KH\left(\frac{aat}{2ee}\right) :: LG \times GK$ or $\overline{CK}^2 - \overline{CG}^2$ $\left(tt - ss + \frac{2esx}{a} - \frac{eexx}{aa}\right) : \overline{GM}^2$ $(yy + xy - 2ay + \frac{1}{4}xx - ax + aa.)$ From whence substituting $4ee - \frac{4ccee}{aa}$ and $-\frac{2ae-2be}{a}$ instead of $tt - ss$ and s, and afterwards multiplying the Extremes and Means, and dividing both Sides by $2t$, we shall get again the propos'd Equation. Therefore, &c. * Art. 55, and 41.

SCHOLIUM.

326. If $ss + 4ee$ should be equal or less than $\frac{4ccee}{aa}$, then it is evident, that the Value of t will be nothing or imaginary; in which Case it will be impossible to construct the Ellipsis that ought to be the Locus of the given Equation. And since this Equation necessarily contains Conditions, therefore it is possible for the same not to have any Line for the Locus thereof; that is, all the Values of y answering to all the Values of x, *may be* imaginary.

*Art. 322. This will appear plain in the general Formula * of the Lemma, which, by transposing some of the Terms, will become $yy - \frac{2n}{m}xy - 2ry + \frac{nn}{mm}xx + \frac{2nr}{m}x + rr = \frac{ptt - pss}{2t} + \frac{2pesx}{2mt} - \frac{eepxx}{2mmt}$, in which Equation the first Member (or Side) is the Square of $y - \frac{n}{m}x - r$; and the second, the Square of t *minus* the Square of $s - \frac{ex}{m}$, multiply'd by the Fraction $\frac{p}{2t}$. Now it is evident, if the Value of the Square tt be nothing or negative; then the Value of the second Member of the Equation will be negative; and so in both Cases we shall have a Square, *viz.* the first Member, having a negative Value, which cannot be.

EXAMPLE II.

327. IT is requir'd to find the Locus of this Equation $yy + \frac{b}{a}xy + xx + cy + fx - ag = 0$, in which I suppose (according to Article 323.) that $\frac{bb}{4aa}$ is less than the Fraction $\frac{1}{1}$, or 1 multiplying the Square xx; *viz.* that b is less than $2a$.

*Art. 322. By comparing the Terms of the general Formula * with those answering to them in the Equation propos'd, we have 1. $\frac{2n}{m} = -\frac{b}{a}$; whence making $m = a$, and then n will be $= -\frac{1}{2}b$. 2. $\frac{nn}{mm} + \frac{eep}{2mmt} = 1$; from whence putting a, $-\frac{1}{2}b$ for m, n, and we shall have $\frac{p}{t} = \frac{4aa - bb}{2ee}$: And so $p = \frac{4aat - bbt}{2ee}$. 3. $r = -\frac{1}{2}c$. 4. $s = \frac{bce - 2afe}{4aa - bb}$. 5. $t = \sqrt{ss + \frac{ccee + 4agee}{4aa - bb}}$. From whence arises the following Construction.

FIG. 173. In the indefinite right Line AP, assume AB $(m) = a$, and draw the right Lines $BE = \frac{1}{2}b = -n$, $AD = \frac{1}{2}c = -r$, parallel to PM, and both on the contrary Side; also through the Point A draw the right Line AE (e) which is given, and through the Point D the right Line DG parallel to AE; in which assume DC $(s) = \frac{bce - 2afe}{4aa - bb}$ from D towards PM, if bc exceeds $2af$, (as it is here suppos'd to do) and the contrary way, if the same be less; then on both Sides the Point C, assume

assume CK and CL, each equal to $t = \sqrt{ss + \frac{ccee + 4agee}{4aa - bb}}$. This being done, describe * an Ellipsis with the Diameter LK $(2t)$ whose Ordinates are parallel to PM, and Parameter the Line KH $(p) = \frac{4aat - bbt}{2ee}$. I say, the Portion OR of this Ellipsis will be the Locus of the proposed Equation. * Art. 162.

For drawing the Line MP from any Point M thereof, making the Angle APM with MP either given or assum'd at pleasure, and meeting the Parallels AE, LK, in the Points F, G; then we have $PF = \frac{bx}{2a}$, and AF or $DG = \frac{ex}{a}$; from whence we get MG or $MP + PF + FG = y + \frac{bx}{2a} + \frac{1}{2}c$, and $CG = \frac{ex}{a} - s$, or $s - \frac{ex}{a}$. But by the Property * of the Ellipsis LK $(2t) : KH \left(\frac{4aat - bbt}{2ee}\right) :: LG \times GK \left(tt - ss + \frac{2esx}{a} - \frac{eexx}{aa}\right) : \overline{GM}^2 \left(yy + \frac{b}{a}xy + cy + \frac{bbxx}{4aa} + \frac{bc}{2a}x + \frac{1}{4}cc\right)$. from whence, (if $\frac{ccee + 4agee}{4aa - bb}$ and $\frac{bce - 2afe}{4aa - bb}$ be substituted for $tt - ss$ and s, and then you multiply the Means and Extremes, and divide by $2t$) and there will arise the proposed Equation. * Art. 55. and 41.

Here it is necessary to observe, that if the Angle AEB be a right one, then will the Angle CGM be so likewise, and the Diameter LK $(2t)$ equal to the Parameter $KH \left(\frac{4aat - bbt}{2ee}\right)$, because $ee = aa - \frac{1}{4}bb$, since AEB is a right-angled Triangle. Therefore, in this Case, the Ellipsis will be a Circle, and the right Line CK or CL $(t) = \sqrt{ss + \frac{1}{4}cc + ag}$, will be a Diameter thereof, and DC (s) will be $= \frac{bc - 2af}{4e}$; from whence arises a much simpler Construction.

Example III.

328. Let it be requir'd to construct the Locus of this Equation $yy + xx - ax = 0$.

By comparing the Terms of the general * Formula with those answering to them in the given Equation, we have 1. $\frac{2n}{m} = 0$, because the Term xy being wanting, the same must be conceiv'd as multiply'd by 0; from whence we get $n = 0$: and therefore $m = e$. 2. $\frac{nn}{mm} + \frac{eep}{2mmt} = 1$; that is, $\frac{p}{2t} = 1$, by substituting 0 and e for n and m; * Art. 322.

and therefore $p = 2t$. 3. $r = o$; because the unknown Quantity y not being found of one Dimension in the given Equation, may be suppos'd also as being multiply'd by o; therefore, if all the Terms where-

* *Art.* 322. in are $\frac{n}{m}$ and r in the general Formula * be struck out, and m and 1 be substituted for e and $\frac{p}{2t}$, then that Formula will become this, *viz.* $yy + xx - 2sx - tt + ss = o$, whose Terms remain to be compared with those of the propos'd Equation. 4. $2s = a$; and therefore $s = \frac{1}{2}a$. 5. $ss - tt = o$; because there is no known Term in the given Equation: and therefore $tt = ss = \frac{1}{4}aa$; and so $t = \frac{1}{2}a$, by extracting the square Root of both Sides. Now the said Values being thus found, the Locus may be constructed after the following manner.

FIG. 174. Because $BE\,(n) = o$, therefore AE falls in AP, and AP in DG likewise, since $AD\,(r) = o$; so that the Point D falls in A. Therefore in AP assume $AC\,(s) = \frac{1}{2}a$ towards PM; as also CK, CL, on both Sides the Point C, each equal to $t = \frac{1}{2}a$ (the Point L here coincides with A;) and with the Diameter AK, whose Ordinates are pa-

* *Art.* 161. rallel to PM, and Parameter the Line $KH\,(p) = 2t = a$, describe * an Ellipsis, and the same shall be the Locus sought.

For if the right Line MP be drawn through any Point M thereof, making the Angle APM with AP either given or taken at pleasure; then we shall have * $AK\,(a) : KH\,(a) :: AP \times PK\,(ax - xx) : \overline{PM}^2\,(yy)$. From whence arises $yy + xx - ax = o$.

* *Art.* 55, *and* 41.

If the Angle APM be a right Angle, then the Ellipsis will become a Circle, and the Line $AK = a$ will be a Diameter thereof.

SCHOLIUM.

329. THERE may happen two Cases, wherein the Locus of a given Equation is a Circle.

Case 1. When the Squares yy and xx are both found with the same Signs and without a Fraction, as also the Plane xy in a given Equation; and when the Angle AEB is a right one (which happens when AF being drawn perpendicular on PM, the Ratio of PF to AP, being the same as the Ratio of BE to AB, is expressed by one half of the Fraction multiplying the Plane xy): Then the Locus of that Equation will be always a Circle, as appears already in *Art.* 324, and the Reason thereof is evident by the general Formula. For comparing the correspondent Terms affected with xx, and we shall have this Equation $\frac{nn}{mm} + \frac{eep}{2mmt} = 1$; and so $\frac{p}{2t} = \frac{mm - nn}{ee} = 1$, since the Triangle AEB being right-angled, the Square $mm = nn + ee$. But

 because

because the Angle AEB is a right one, the Angle CGM being that made by the Diameter LK and the Ordinates thereof, will be a right one also; and so since the Diameter LK is equal to its Parameter KH, the Ellipsis will then become a Circle.

Case 2. When the Squares yy and xx, are both found with the same Signs, and without a Fraction, in an Equation, wanting the Plane xy, and when the Angle APM is a right one: Then the Locus of that Equation will be always a Circle, as hath been observ'd in *Art.* 328; and this may be proved by means of the general Formula. For because the Plane xy is not in the propos'd Equation, the Fraction $\frac{2n}{m}$ in that Formula will be $= 0$; and therefore $BE\ (n) = 0$, and $m = e$. From whence it follows, 1. that the Diameter LK is parallel to the Line AP, and so the Angle CGM made by the same and its Ordinates, being equal to the Angle APM, will be a right Angle. 2. That the Fraction $\frac{ee}{mm} + \frac{eep}{2mmt}$ which multiplies the Square xx in the Formula, becomes $\frac{p}{2t}$, from whence we have $\frac{p}{2t} = 1$: That is, the Diameter LK will be equal to its Parameter KH. Therefore the Ellipsis which is the Locus of the given Equation, will be a Circle. And because the general Formula in this Case, is

$$\begin{aligned} yy + xx - 2ry - 2sx + rr &= 0 \\ -tt& \\ +ss& \end{aligned}$$

You may abridge the Calculus by comparing the Terms of this Formula, with those of the proposed Equation, and finding by that means the Values of r, s, t, which serve to describe the Circle that is the Locus of the proposed Equation.

The FUNDAMENTAL LEMMA *for the Construction of Loci, which are Hyperbola's respecting their Diameters.*

330. THE same things being laid down as in the foregoing Lemma for the Ellipsis. With the Diameter LK $(2t)$, whose Ordinates are parallel to PM, and Parameter KH (p) describe * an Hyperbola or two opposite Sections. I say the Portion or Portions (OM) thereof, contain'd in the Angle PAD made by the Line AP and the Line AD drawn through the fixed Point A parallel to PM towards the same Parts, will be the Locus of the following Equation or Formula. Fig. 175. 176.

* Art. 161.

$yy-$

$$\begin{array}{l} yy - \frac{2n}{m}xy + \frac{nn}{mm}xx - 2ry + \frac{2nr}{m}x + rr = 0 \\ \qquad\quad - \frac{eep}{2mmt} \qquad\qquad\quad + \frac{2eps}{2mt}x \pm \frac{ptt}{2t} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad - \frac{pss}{2t} \end{array}$$

in which you must observe that $\frac{ptt}{2t}$ is affirmative, when LK is a first Diameter, and negative when the same is a second.

For if the Line MP be drawn through any Point M of that Portion, making the Angle APM, with AP, either given or taken at pleasure, and meeting the Parallels AE, DG, in the Points F, G; then by the
* Art. 81, and 118. Property * of the Hyperbola, we shall have $KL\,(2t) : KH\,(p) :: \overline{CG}^2 \pm \overline{CK}^2 \left(\frac{eexx}{mm} - \frac{2esx}{m} + ss \pm tt\right) : \overline{GM}^2 = \frac{peexx}{2mmt} - \frac{2epsx}{2mt} + \frac{pss}{2t} \pm \frac{ptt}{2t} = yy - \frac{2n}{m}xy - 2ry + \frac{nn}{mm}xx + \frac{2nr}{m}x + rr = 0$. Therefore, &c.

If the Diameter $KL\,(2t)$ and its Parameter $KH\,(p)$ be equal between themselves; then the Hyperbola will be an equilateral one.

COROLLARY.

331. It is manifest, 1. That the two Squares yy and xx have always different Signs in that Formula, when the Plane xy happens not to be therein; or even when it is in the same, if $\frac{eep}{2mmt}$ exceeds $\frac{nn}{mm}$. 2. That the said Squares may have the same Signs, upon Condition that the Plane xy be not in the Formula, and the Square $\left(\frac{nn}{mm}\right)$ of one half the Fraction multiplying xy, be greater than the Fraction $\frac{nn}{mm} - \frac{eep}{2mmt}$ multiplying the Square xx.

PROPOSITION IV.

Problem.

332. *To construct the Locus of a given Equation, wherein the Squares* yy *and* xx *have different Signs, or even the same Signs, upon Condition that the Plane* xy *does not happen therein, and the Square of one half the Fraction multiplying the said Plane* xy*, be greater than the Fraction multiplying the Square* xx*. Supposing the Square* yy *to be free from Fractions.*

The

The Locus of the Equation being an Hyperbola, muſt be conſtructed in the ſame manner as the Ellipſis was in the laſt Problem, as ſhall appear in the following Examples.

Example I.

333. Let there be an Equation $yy + \frac{2b}{a}xy + \frac{f}{a}xx + 2cy - 2gx - bb = o$, whoſe Locus it is requir'd to conſtruct, ſuppoſing the Square $\frac{bb}{aa}$ therein to exceed $\frac{f}{a}$.

Compare the Terms of this Equation with thoſe anſwering them in the Formula of the Lemma, and then we have 1. $\frac{2n}{m} = -\frac{2b}{a}$, and therefore if you make $m = a$, then n will be $= -b$. 2. $\frac{eep}{2mmt} - \frac{nn}{mm} = \frac{f}{a}$, and ſo $\frac{p}{2t} = \frac{bb - af}{ee}$, and $p = \frac{2bbt - 2aft}{ee}$. 3. $r = c$. 4. $\frac{2nr}{m} + \frac{2eps}{2mt} = -2g$, from whence ſubſtituting for $m, n, r, \frac{p}{2t}$, their Values already found, and we ſhall get $s = \frac{-bce - age}{bb - af}$. 5. $\mp tt = ss - \frac{2rrt - 2bbt}{p} = ss - \frac{eecc - eebb}{bb - af}$, tt being affirmative when the Square ss exceeds $\frac{eecc + eebb}{bb - af}$, and negative when the ſame is leſs, becauſe the Square tt muſt be poſitive; from whence there are two Caſes. Now the Values of m, n, r, s, t, p, being thus determined, the Locus of the Equation may be conſtructed after the following manner, uſing the Conſtruction of the Lemma.

In AP, aſſume $AB = a$, and draw the right Lines $BE = b = -n$, $AD = c = -r$, parallel to PM, and on the coutrary Side of AP in reſpect to PM; alſo through the Points A, E, draw the right Line AE (e) which is given, and through the Point D the indefinite right Line DG parallel to AE, in which aſſume $DC = \frac{eag + ebc}{bb - af} = -s$ from D towards PM, and on both Sides the Point C aſſume CL, CK, each equal to $t = \sqrt{ss - \frac{eecc - eebb}{bb - af}}$ or $\sqrt{\frac{eecc + eebb}{bb - af} - ss}$, according as ss is greater or leſs than $\frac{eecc + eebb}{bb - af}$. This being done, with the Diameter LK, whoſe Ordinates are parallel to PM, and Parameter the Fig. 178.

Line $KH\ (p) = \frac{2bbt - 2aft}{ee}$, describe an Hyperbola. observing that LK must be a first Diameter (*Fig.* 177.) in the former Case, and a second (*Fig.* 178.) in the latter Case. Then the Portion OM of that Hyperbola will be the Locus sought.

For if from any one of the Points thereof, as M, the Line MP be drawn parallel to AD, meeting the Lines AB, AE, DG, in the Points P, F, G; then we shall have $PF = \frac{bx}{a}$, and AF or $DG = \frac{ex}{a}$. And consequently $MG = y + \frac{bx}{a} + c$, and CG or $DG + CD = \frac{ex}{a} - s$, because $CD = -s$. But by the Property of the Hyperbola, $LK\ (2t) : KH \left(\frac{2bbt - 2aft}{ee}\right) :: \overline{CG}^2 \pm \overline{CK}^2 \left(\frac{eexx}{aa} - \frac{2esx}{a} + ss \pm tt\right) : \overline{GM}^2 \left(yy + \frac{2b}{a}xy + 2cy + \frac{bb}{aa}xx + \frac{2bc}{a}x + cc\right)$ which gives the given Equation by substituting $\frac{ccee - eebb}{bb - af}$ and $\frac{-bce - age}{bb - af}$) for their Values $ss \pm tt$ and s, and multiplying the Means and Extremes, and dividing by $2t$. Therefore, &c.

SCHOLIUM.

334. IF ss be $= \frac{ccee + eebb}{bb - af}$; then it is manifest, that the Value of tt will be $= o$, and so the Construction of the Hyperbola will be impossible. In which Case it must be observ'd, that the proposed Equation may always be brought lower, so that the Locus thereof, which ought to be an Hyperbola, or two opposite ones, will become one or two right Lines: For in our Example the proposed Equation may be reduced to this Proportion, *viz.* $ee : bb - af :: \frac{eexx}{aa} - \frac{2esx}{a} + ss \pm tt : yy + \frac{2b}{a}xy + \frac{bb}{aa}xx + 2cy + \frac{2bc}{a}x + cc$; from whence striking out $tt = o$, multiplying the Extremes and Means, and extracting the square Root of both Sides, and there will arise $ey + \frac{ebx}{a} + ec = \frac{ex}{a} - s\sqrt{bb - af}$, that is, (putting $\frac{bce + age}{bb - af}$ for $-s$, and dividing both Sides by e) this Equation $y + \frac{bx}{a} + c = \frac{x\sqrt{bb - af}}{a} + \frac{ag + bc}{\sqrt{bb - af}}$, or $y = \frac{-b + \sqrt{bb - af}}{a}x$ +

$\frac{ag+bc}{a\sqrt{bb-af}} - c$, which will be chang'd into this $y = p - \frac{n}{m}x$ (by making $\frac{n}{m} = \frac{b-\sqrt{bb-af}}{a}$, and $p = \frac{ag+bc}{\sqrt{bb-af}} - c$) whose Locus is a strait Line, and may be constructed by *Art.* 306.

The Reason of this is evident by the general Formula of the Lemma: For if the Term $\mp\frac{ptt}{2t}$, be struck out in that Lemma, because $tt = o$; then by transposing some Terms, and extracting the square Root, we shall have this Equation $y - \frac{n}{m}x - r = \frac{ex}{m} - s\sqrt{\frac{p}{2t}}$ or $s - \frac{ex}{m}\sqrt{\frac{p}{2t}}$, wherein the unknown Quantities x and y, are but of one Dimension, and consequently the Locus thereof will be strait Lines.

EXAMPLE II.

335. THE Locus of the following given Equation is requir'd, *viz.* $yy - xx + 2ay + ax = o$.

By comparing the correspondent given Terms, we have 1. $\frac{2n}{m} = o$, because the Plane xy is not in the proposed Equation; from whence we get $n = o$, and so $m = e$. 2. $\frac{p}{2t} = 1$, and therefore $p = 2t$. 3. $r = -a$. 4. $\frac{2ps}{2t} = a$; whence $s = \frac{1}{2}a$. 5. $rr \mp \frac{ptt}{2t} - \frac{pss}{2t} = o$; and so $\mp tt = ss - \frac{2rrt}{p} = -\frac{3}{4}aa$, by putting $-a$, 1, $\frac{1}{2}a$, for their Values r, $\frac{2t}{p}$, s; from whence we know, that $-tt$ must be taken in the last Term in the Formula, and not $+tt$, that so the Value of tt may be positive. Now the Locus may be thus constructed.

Because $AD\ (r) = -a$, therefore through the Point A draw the right Line $AD = a$, parallel to PM, and on the contrary Side of AP with regard to PM; and because $BE\ (n) = o$, therefore through the Point D draw the right Line DG parallel to AP, in which assume $DC\ (s) = \frac{1}{2}a$ from D towards PM; also assume CK, CL, on both Sides the Point C, each equal to $t = \sqrt{\frac{3}{4}aa}$. This being done with the second Diameter LK (because $-tt$ was taken in the last Term of the Formula) whose Ordinates are parallel to PM, and Parameter the right Line $KH\ (p) = 2t = LK$, describe an Hyperbola. Then the Portion OM thereof will be the Locus sought. FIG. 179.

For if MP be drawn thro' any one of its Points M parallel to AC; meeting

 the

the right Lines AP, DG, in the Points P,G; then we ſhall have $MG = y + a$, CG or $DG - DC = x - \frac{1}{2}a$; and by the Property of the Hyperbola, $LK\ (2t) : KH\ (2t) :: \overline{CG}^2 + \overline{CK}^2\ (xx - ax + \frac{1}{4}aa + tt) : \overline{GM}^2\ (yy + 2ay + aa)$; which gives the propoſed Equation $yy + 2ay - xx + ax = 0$, by putting $\frac{1}{4}aa$ for tt. Hence it appears, that the Hyperbola is an Equilateral one.

SCHOLIUM.

336. WHEN the Squares yy and xx happen to have different Signs, and without Fractions in an Equation, wherein the Plane xy is not; then the Locus of that Equation will be always an Equilateral Hyperbola: for the Fraction $\left(\frac{2n}{m}\right)$ of the Formula will be $= 0$; and therefore $BE\ (n) = 0$, and $m = e$. Conſequently the Fraction $\frac{nn}{mm} - \frac{eep}{2mmt}$, which multiplies the Square xx in the Formula, will become $-\frac{p}{2t}$; and ſo we ſhall have $-\frac{p}{2t} = 1$, that is, the Diameter LK will be equal to its Parameter KH; or, which is the ſame thing, the Hyperbola will be an Equilateral one. And becauſe the general Formula will then be changed into this;

$$\begin{aligned} yy - xx - 2ry + 2sx + rr &= 0 \\ &+ tt \\ &- ss \end{aligned}$$

therefore the Values of r, s, t, may be firſt gotten from the ſame, and afterwards the Equilateral Hyperbola being the Locus of the propoſed Equation may be conſtructed by means of thoſe Values; which will very much ſhorten the Operation.

The FUNDAMENTAL LEMMA, *for the Conſtruction of the Loci of Equations, which are Hyperbola's between their Aſymptotes.*

337. LET there be (as in *Def.* 1.) two unknown and indeterminate right Lines $AP\ (x)$, PM, y, making the Angle APM with one another either given or taken at pleaſure; alſo let m, n, p, r, s, be given right Lines. This being ſuppos'd;

FIG. 180. 1. In the Line AP, aſſume $AB = m$, and draw the right Lines $BE = n$, $AD = r$, parallel to PM, and both on the ſame Side AP with regard to PM; then through the Point A draw the right Line $AE\ (e)$ which is given, and through the Point D the indefinite right Line DG parallel to AE; in which aſſume $DC = s$, $CK = e$, both tending

tending from D the same way as AP tends, and draw the indefinite right Line CL parallel to PM, and on the same Side AP; and lastly draw the Line $KH = p$. Then describe * an Hyperbola between the Asymptotes CL, CK, passing through the Point H. I say, the same will be the Locus of the following Equation or Formula, *viz.* *Art. 130, 131.

$$xy - \frac{n}{m}xx - \frac{ms}{e}y + \frac{ns}{e}x + \frac{mrs}{e} = 0$$
$$- rx - mp$$

For $GM = y - \frac{nx}{m} - r$, $CG = \frac{ex}{m} - s$, and by the Property of the Hyperbola * $CG \times GM \left(\frac{exy}{m} - sy - \frac{enxx}{mm} + \frac{nsx}{m} - \frac{erx}{m} + rs\right) = CK \times KH\,(ep)$; from whence freeing the Term xy from Fractions, and putting all the Terms in order, there will arise the same Equation as above, *viz.* $xy - \frac{n}{m}xx - \frac{ms}{e}y$ &c. *Art. 101.

2. Through the fixed Point A draw the indefinite Line AQ parallel to PM and on the same Side AP, with regard to PM; then in AQ assume $AB = m$, draw $BE = n$ parallel to AP, and towards the same Parts, and join the determinate Points A, E, by the Line $AE\,(e)$; moreover in AP assume $AD = r$ from A towards PM, and draw the indefinite Line DG parallel to AE, in which assume $DC = s$, $CK = e$ tending the same way as PM tends, and draw the indefinite Right Lines CL and $KH = p$ both parallel to AP, (on the same Side thereof as PM). This being done between the Asymptotes CL, CK, describe * an Hyperbola which passes through the Point H. Then I say that the same will be the Locus of the following Equation or Formula, *viz.* Fig. 181. *Art. 130, 131.

$$xy - \frac{n}{m}yy - \frac{ms}{e}x + \frac{ns}{e}y + \frac{mrs}{e} = 0$$
$$- ry - mp$$

For if the Line MQ be drawn from any Point M thereof, parallel to AP, and meeting the Parallels AE, DG, in the Points F, G; then the similar Triangles ABE, AQF, will give these Proportions $AB\,(m) : AE\,(e) :: AQ$ or $PM\,(y) : AF$ or $DG = \frac{ey}{m}$, and $AB\,(m) : BE\,(n) :: AQ\,(y) : QF = \frac{ny}{m}$. And consequently $GM = x - \frac{ny}{m} - r$, $CG = \frac{ey}{m} - s$. But by the Property of the Hyperbola, $CG \times GM = CK \times KH$; from whence, by substituting for these Lines their analytical Values, and freeing the Term xy from Fractions, the second Formula aforegoing will be had. Therefore, &c.

COROLLARY.

338. IT is manifest, 1. That the Term xy is found always in both the Formula's; since the same being multiply'd by no Fraction, cannot be supposed equal to nothing, in order for that Term to vanish or be struck out. 2. In either of the Formulas there is only one of the Squares xx and yy, which will vanish if the Fraction $\frac{n}{m}$ multiplying the same be equal to nothing.

PROPOSITION V.

Problem.

339. *TO find the Locus of a given Equation, wherein is the Plane* xy, *without either of the Squares* xx, yy, *or only with one of the Squares* xx *or* yy.

Free the Plane xy from Fractions, and compare the Terms of the given Equation with those that answer to it in the first Formula, when the Square xx is in the given Equation, and with those of the second Formula, when yy is in the same; and finally either of them at pleasure, when neither of the Squares xx and yy are in the given Equation. By this Means, get the Values of the Quantities m, n, p, r, s, and then by means of those Values describe an Hyperbola between its Asymptotes, according to the Directions in the foregoing Lemma, always observing to draw the Lines whose Values are negative, on the contrary Side of AP, with respect to PM, and assume those whose Values are so likewise, tending the contrary way to which AP tends. The following Example will make this manifest.

EXAMPLE I.

340. IT is requir'd to find the Locus of $xy - \frac{b}{a}xx - cy = o$.

Because the Square xx happens to be in the given Equation, therefore chuse the first Formula, and then comparing the Terms thereof with those of the proposed Equation, and we have 1. $\frac{n}{m} = \frac{b}{a}$, whence making $m = a$, and we get $n = b$. 2. $\frac{ms}{e} = c$, and therefore $s = \frac{ec}{a}$. 3. $\frac{ns}{e} - r = o$, because x is not in the given Equation, and therefore $r = \frac{bc}{a}$. 4. $\frac{mrs}{e} - mp = o$, because there is no Term in the proposed Equation quite known: And therefore $p = \frac{rs}{e} = \frac{bcc}{aa}$. Now because the

Values

Values of $AP(m)$, $BE(n)$, $CD(s)$, $AD(r)$, $KH(p)$ are all positive, the Locus must be constructed in the very same manner as in the Lemma (*Fig.* 180.) observing to use the Values of the Lines as here determin'd.

For $GM = y - \frac{bx}{a} - \frac{bc}{a}$, CG or $DG - DC = \frac{ex - ec}{a}$, and by the Property of the Hyperbola $CG \times GM = CK \times KH$, that is, by substituting the analytick Values, the proposed Equation. Therefore, &c. FIG. 180.

EXAMPLE II.

341. LET $xy + \frac{b}{a}yy - cy - ff = o$, be an Equation, whose Locus it is required to construct.

Because the Square yy is in this Equation, therefore chuse the second Formula, and by comparing the Terms thereof with those of the proposed Equation, we have, 1. $\frac{n}{m} = -\frac{b}{a}$; and making $m = a$, we get $n = -b$. 2. $\frac{ms}{e} = o$, and therefore $s = o$. 3. $r = c$. 4. $mp = ff$, and therefore $p = \frac{ff}{a}$. From whence arises the following Construction.

Through the fixed Point A draw the indefinite Right Line AQ parallel to PM, and on the same Side AP, and in AQ assume AB $(m) = a$, and draw $BE = b = -n$ parallel to AP, and tending the contrary way to AP; also through the determinate Points A, E, draw the Line $AE(e)$. This being done, in AP, take $AD\ (r) = c$ from A towards PM, and draw the indefinite Right Line DG parallel to AE, and because the Points D, C, do coincide, since $DC(s) = o$, therefore in DG assume $DK = e$, tending the same way as PM, and draw the Line $KH(p) = \frac{ff}{a}$, parallel to AP, (and tending the same way) as also the indefinite Right Line DL, which here falls in AP; then describe an Hyperbola between the Asymptotes DL, DK, which passes through the Point H. And the same will be the Locus requir'd. FIG. 182.

For if the Right Line MQ be drawn from any Point M thereof, parallel to AP, and meeting the Parallels AE, DG, in the Points F, G; then we shall have GM or $MQ + QF - FG = x + \frac{by}{a} - c$, DG or $AF = \frac{ey}{a}$, and therefore $DG \times GM = \frac{exy}{a} + \frac{ebyy}{aa} - \frac{ecy}{a} = DK \times KH\left(\frac{eff}{a}\right)$. From whence by freeing the Term xy from Fractions, there

will

will arise the proposed Equation $xy + \frac{b}{a}yy - cy - ff = o$.

SCHOLIUM.

342. IF for the arbitrary Quantity AB (m) be taken some Value greater or less than a, then the Values of CK (e) and KH (p) will alter, but the Values of the Rectangle $CK \times KH$ (ep) and of the Right Lines AD (r), CD (s) will remain always the same: For in the Expressions of these Values are contain'd only the Ratio's $\frac{n}{m}$, $\frac{n}{e}$, $\frac{m}{e}$ which do not vary, because in the Triangle ABE, the Angle ABE is given, as also the Ratio $\frac{n}{m}$ (which in this Example is $\frac{b}{a}$) of the Side AB (m) to the Side BE (n). And because the Hyperbola which
*Art. 101. must pass through the Point H, shall be the same * always, let the Magnitude of CK (e) and KH (p), be what it will, provided the Rectangle $CK \times KH$ remains the same, therefore the same Hyperbola will be constructed, let the Magnitude of the arbitrary Quantity AB (m) be what it will.

EXAMPLE III.

343. IT is requir'd to construct the Locus of the given Equation $xy - ay + bx + cc = o$.

Because neither of the Squares xx, yy, are found in the proposed Equation, therefore chuse indifferently any one of the two Formulas, for Example, the first, and comparing the Terms of the same with those of the proposed Equation, we have 1. $\frac{n}{m} = o$, and therefore $n = o$, and $m = e$, and make $m = a$. 2. $\frac{ms}{e}$ or $s = a$. 3. $r = -b$, because $\frac{ns}{e} = o$. 4. $rs - mp = cc$, and therefore $p = -b - \frac{cc}{a}$. Now the Values of m, n, r, s, p, being thus determin'd, the Locus of the proposed Equation may be constructed after the following manner

FIG. 183. Because AD (r) $= -b$, therefore draw the Line $AD = b$ parallel to PM, and on the contrary Side of AP with regard to PM; and because BE (n) $= o$, therefore draw the indefinite right Line IG parallel to AP, and in the same assume DC (s) $= a$, CK (e) $= m = a$ tending the same way as AP: Also draw the indefinite right Line

4 CL,

CL, and the Line $KH = b + \frac{cc}{a} = -p$, both parallel to PM, and tending the contrary way. Then describe an Hyperbola opposite to that, whose Asymptotes are CL, CK, and which passes thro' the Point H. I say, the indefinite Portion OM thereof contain'd in the Angle PAS, form'd by the indefinite right Line AP, and the Line AS, drawn parallel to PM on the same Side AP, shall be the Locus sought.

For GM or $PG + PM = y + b$, and CG or $CD - DG = a - x$, and consequently $CG \times GM = ay - xy + ab - bx = CK \times KH$ $(ab + cc)$; from whence, striking out the Rectangle ab from both Sides, and transposing, there arises $xy - ay + bx + cc = 0$, which is the propos'd Equation. The Description of the Hyperbola passing through the Point H in this Example is useless, because no Point thereof can fall in the Angle PAS, wherein the Points M must be suppos'd to fall.

SCHOLIUM.

344. IN comparing the Terms of the Formula with those of a given Equation, if it should happen that $p = 0$; then it would be impossible to describe the Hyperbola that ought to be the Locus thereof, because its *Power*, which is equal to the Rectangle pe, will be $= 0$. But then the Equation may be brought lower, so that the Locus will become a strait Line. For Example; if the Term mp be struck out of the first Formula of the Lemma, then that Formula will become $xy - \frac{n}{m}xx - \frac{ms}{e}y + \frac{ns}{e}x - rx + \frac{mrs}{e} = 0$, which being divided by $\frac{ex}{m} - s$ gives $y - \frac{nx}{m} - r = 0$, whose Locus is * a strait Line. *Art. 306.

PROPOSITION VI.

Problem.

345. *TO construct the Locus of any given Equation of the second Degree.*

Bring over all the Terms of the Equation to one Side, so that one Member thereof be 0, then there may happen two Cases.

Case 1. When the Plane xy is not in the given Equation. 1. If there be but one of the Squares yy or xx therein, then the Locus will be * a Parabola. 2. If both the Squares yy and xx are found therein with the same Signs, then the Locus will be * an Ellipsis or Circle. 3. If the said two Squares are found therein with different Signs, then the Locus thereof will be * an Hyperbola, or the opposite Sections, regarding their Diameters. *Art. 310. *Art. 324. *Art. 332.

Case 2. When the Plane xy happens to be in a given Equation. 1. If neither of the Squares yy and xx, or but one of them, are found in the Equation, then the Locus will be * an Hyperbola between its Asymptotes. 2. If the Squares yy and xx are found therein with different Signs, then the Locus shall be * an Hyperbola regarding its Diameters. 3. If the said two Squares have the same Signs, the Square yy must be freed from Fractions, and then the Locus shall be * a Parabola, when the Square of half the Fraction multiplying xy, is equal to the Fraction multiplying xx; an Ellipsis * or Circle, when the same is less; and finally, an Hyperbola, * or two opposite ones, regarding their Diameters, when the same is greater.

* *Art.* 339. * *Art.* 332. * *Art.* 310. * *Art.* 324. * *Art.* 332.

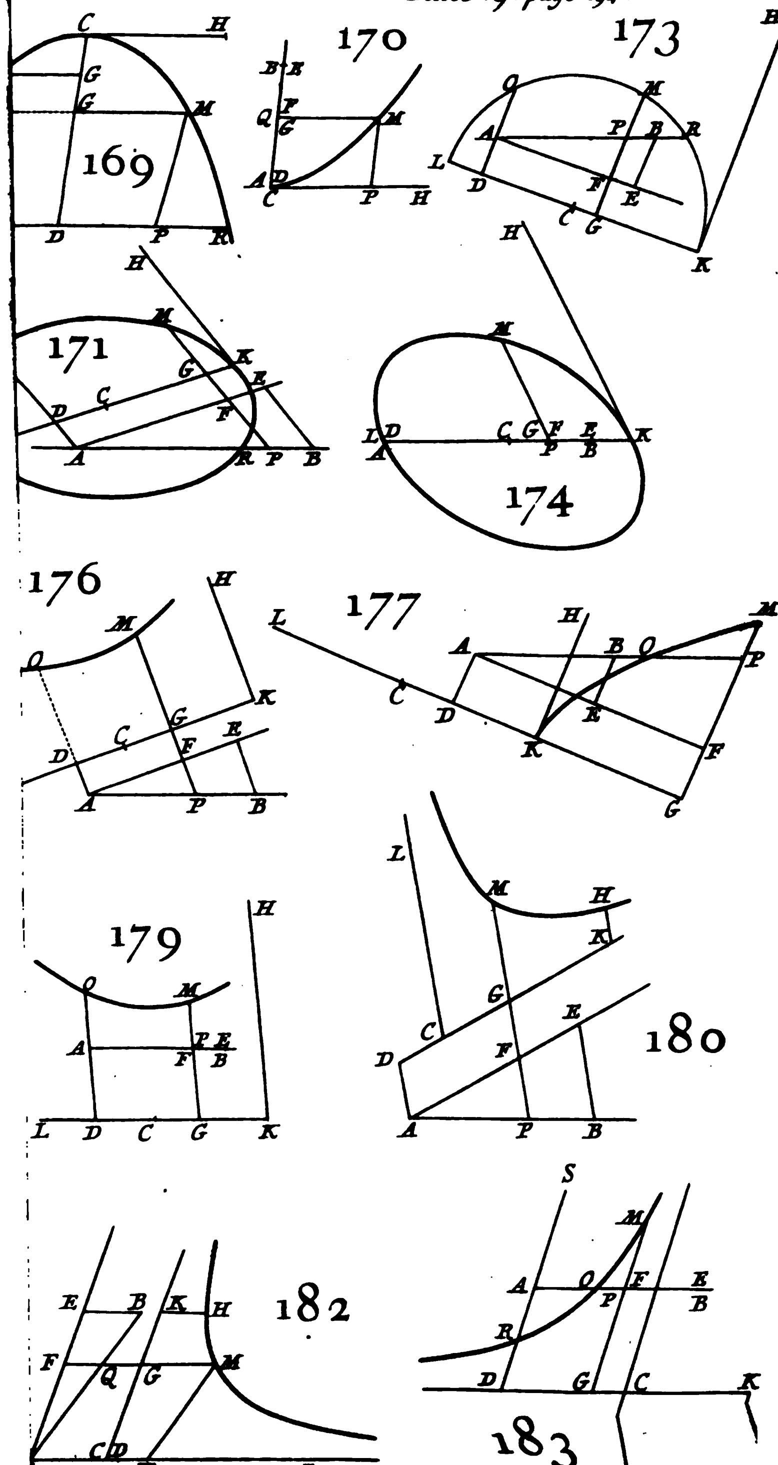
Now describe the Locus, by *Art.* 310. if the same be a Parabola; by *Art.* 324. if the same be an Ellipsis or Circle; by *Art.* 332. if it be an Hyperbola, or the opposite Sections regarding their Diameters; and lastly, by *Art.* 339. if the same be an Hyperbola between the Asymptotes. All this is but a Continuation of those four Articles.

COROLLARY.

346. BEcause the Conick Section found by the aforesaid Rules, is the Locus * of all the positive and negative Values of the unknown Quantity y, answering to the positive and negative Values of x, in a given Equation of the second Degree; therefore there is but that Section only that can be the Locus of the given Equation.

* *Art.* 314.

The End of the Seventh Book.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BOOK VIII.

A General PROPOSITION.

347. *TO find the Locus of an infinite Number of Points, having all certain known Conditions; provided the said Locus exceeds not the second Degree.* FIG. 184.

1. Suppose two unknown and indeterminate Right Lines AP (x) PM (y) making an Angle (APM) with one another, either given or taken at pleasure, as being known and determinate; and let one of them as AP, have a fixed and stable Origin in the Point A, and be extended along a Right Line given in Position; and let the other PM, whose Extremity M determines one of the Points sought, continually alter its Position, and be always parallel to the same Line. 2. Draw the other Lines that you think convenient for the Solution of the Problem, and express them by Letters, *viz*. the known Lines by the former Letters of the Alphabet, and the unknown ones, by the latter Letters. 3. Suppose the Question resolved, and after you have gone over all the Conditions thereof, form an Equation containing only the two unknown Lines x and y, mixed with known ones. 4. Having form'd an Equation, wherein the unknown Lines x and y have not more than two Dimensions, construct the Locus of that Equation according to the Directions in the last Book; and then the Locus thus constructed, will solve the Question. All this will be manifest by the following Examples.

EXAMPLE I.

348. THE Angle BAC being given, to find the Point M in it; such that two right Lines MF, MG, drawn from the same, making the given Angles MFB, MGC, with the Sides AB, AC, always towards the same Parts; the Right Line MF may be to the Right Line MG in a given Ratio, suppose as a to b. And because there are an infinite Number of such Points, it is requir'd to find the Line containing them all, that is, their Locus. FIG. 184.

Through the Point M, which is suppos'd to be one of the Points sought, draw the Line MP parallel to AC, and conceive the two unknown and indeterminate Lines $AP(x)$, $PM(y)$, as known and determinate. Then in the Side AB assume $AB=a$, and draw the Right Lines BC, BD, parallel to MF, MG, meeting the other Side AC (produced if necessary) in the Points C, D. This being done, call the known Lines AC, c; BC, f; BD, g; then drawing MQ parallel to AB, the Triangles ACB, PMF, and ABD, QMG, will be similar, therefore $AC(c):CB(f)::MP(y):MF=\frac{fy}{c}$, and $AB(a):BD(g)::MQ$ or $AP(x):MG=\frac{gx}{a}$; and this satisfies the first State of the Problem, since the Lines MF, MG, are suppos'd to be always parallel to the same two straight Lines BC, BD, which make the given Angles with the Sides AB, AC. Now by the second Condition remaining to be fulfilled, it must be as $MF\left(\frac{fy}{c}\right):MG\left(\frac{gx}{a}\right)::a:b$; from whence there arises $y=\frac{cgx}{bf}$ which includes all the Conditions of the Problem, and the Locus of the same will consequently be the Locus sought, and may be constructed thus *;

* Art. 306.

In the Line AP, assume $AH=b$, and draw $HE=\frac{cg}{f}$ parallel to PM, and on the same Side AP with regard to PM; then if the indefinite Right Line AE be drawn, I say the same will be the Locus of all the sought Points M.

For through any Point M thereof draw the Right Lines MP, MQ, parallel to the two Sides AC, AB, and the right Lines MF, MG, parallel to BC, BD, which consequently shall make the given Angles with the Sides AB, AC; then because the Triangles AHE, APM, are similar, therefore $AH(b):HE\left(\frac{cg}{f}\right)::AP(x):PM(y)=\frac{cgx}{bf}$, and because the Triangles ACB, PMF, and ABD, QMG, are similar, therefore $AC(c):CB(f)::MP\left(\frac{cgx}{bf}\right):MF=\frac{gx}{b}$; and $AB(a):BD(g)::MQ$ or $AP(x)$ $MG=\frac{gx}{a}$. And consequently $MF\left(\frac{gx}{b}\right):MG\left(\frac{gx}{a}\right)::a:b$. Which is the Proportion propos'd.

We have solv'd this Problem by Calculation, only to explain the general Proposition, and shew the Application thereof in beginning first with simple and easy Examples; for the same may be solv'd in a much easier manner, without any Calculus, thus.

Draw

Draw the right Lines AK, AL, making the given Angles KAE, LAC, with AB, AC, being to one another in the given Ratio of a to b. Also draw the right Lines KM, LM, parallel to the Sides AB, AC, meeting each other in the Point M; through which, and the Vertex A of the given Angle BAC, draw the Line AM; and the same will be the Locus sought. Fig. 185.

For drawing the right Lines ER, ES, from any Point E thereof parallel to AK, AL; then because the Triangles AER, MAK, and AES, MAL, are similar; therefore $ER : AK :: AE : AM :: ES : AL$. And therefore $ER : ES :: AK : AL :: a : b$.

EXAMPLE II.

349. THE Parallels AB, CD, being given in Position; to find the Locus of all the Points M so situate between those Parallels, that the right Lines MP, MG, being drawn, making always given Angles MPB, MGD, with them towards the same Part; the said Lines MP, MG may be to one another always in the given Ratio of a to b. Fig. 186.

Assume any Point (A) of the Line AB, for the fix'd Origin of the indeterminate Line AP (x), and the two unknown and indeterminate Lines AP (x), PM (y) being suppos'd known and determinate, draw the Lines AC, AE, parallel to MP, MG; then call the known Lines AC, c; AE, f; this being done, produce PM till it meets CD in F; and the similar Triangles CAE, FMG, shall give $AC\,(c) : AE\,(f) :: MF\,(c-y) : MG = \frac{cf-fy}{c}$. But according to the Condition of the Problem remaining to be fulfilled, it must be as $MP\,(y) : MG\,\left(\frac{cf-fy}{c}\right) :: a : b$; from whence arises this Equation $y = \frac{acf}{bc+af}$, which takes in all the Conditions of the Problem, and the Locus thereof, which is * an indefinite right Line HM drawn parallel to AB, so that AH be $= \frac{acf}{bc+af}$, will be the Locus sought. * Art. 307.

EXAMPLE III.

350. TWO Points A and B being given: To find a third Point M such, that if two right Lines MA, MB be drawn to the same, they may be always to one another in the given Ratio of a to b. And because there are an infinite Number of such Points, it is requir'd to describe the Locus of them all. Fig. 187.

Here there may happen three different Cases, according as a is less, greater, or equal to b.

Case

Case 1. From the Point M, which is suppos'd to be one of the Points sought, draw the Line MP perpendicular to AB (for since no Angle is given in the Problem, we chuse a right Angle, as being more simple than any other) and the two unknown and indeterminate right Lines $AP\,(x)$, $PM\,(y)$, being suppos'd known and determinate, call the given Line AB, c; then because the Triangles APM, BPM, are right-angled, therefore $\overline{AM}^2 = xx + yy$, and $\overline{BM}^2 = cc - 2cx + xx + yy$. But according to the Condition of the Problem, it must be as $\overline{AM}^2\,(xx+yy) : \overline{BM}^2\,(cc - 2cx + xx + yy) :: aa : bb$. From whence (multiplying the Extremes and Means; and afterwards dividing by $bb - aa$) we shall get this Equation, *viz.* $yy + xx + \frac{2aac}{bb-aa}x - \frac{aacc}{bb-aa} = 0$, the Locus of which will be that sought; and it may be constructed by means of *Art.* 322. after the following manner.

FIG. 187. In the Line AP assume $AC = \frac{aac}{bb-aa}$ from A the contrary way to P; and about the Centre C, with the Radius CD or $CE = \frac{abc}{bb-aa}$ describe the Circumference of a Circle. I say, the Portion (DMO) thereof contain'd in the Angle PAO, made by the Line AP, and the right Line AO, drawn parallel to PM, and on the same Side AP with regard to PM, will be the Locus of the Equation found as above.

For if MP be drawn from any Point M thereof perpendicular on AB; then, by the Nature of the Circle, we shall have $\overline{CD}^2 - \overline{CP}^2$ or $EP \times PD = \overline{PM}^2$; that is, the precedent Equation, by putting for these Squares their analytick Values.

Now, if the Points M be supposed to fall in the Angle EAR, opposite to the Angle BAO, in which they were supposed to be situate
*Art. 304. in the foresaid Process; then if you make * $AP = -x$, and $PM = -y$, there will arise the same Equation as above, from the Conditions of the Problem and the Property of the Portion (RME) of the Circumference already describ'd; and therefore that Portion is the Locus of all the sought Points M, situate in the Angle EAR. And lastly, if the Points M be supposed to fall in the Angle BAR, and afterwards in the Angle EAO, you will find, in like manner, that the Portions DR, EO, of the same Circumference, will be the Loci of those Points (observing to make $PM = -y$, when the same falls on the other Side the Line AB; and $AP = -x$, when the Point P falls on the other Side of the fixed Point A); therefore the whole Circumference, whose Diameter is the Line DE, is the compleat Locus of all the sought Points M.

Case

Case 2. In reasoning after the same manner as in *Case* 1. there will arise this Equation, *viz.* $yy + xx - \frac{2aacx}{aa-bb} + \frac{aacc}{aa-bb} = o$, whose Locus may be thus constructed.

In AP assume $AC = \frac{aac}{aa-bb}$ from A towards PM; and with the Centre C, and Radius CD, or $CE = \frac{abc}{aa-bb}$ describe a Circle. I say, the Circumference thereof will be the Locus of all the sought Points M. We prove this after the same manner as in *Case* 1. FIG. 188.

The Constructions in the two last Cases may be much shorten'd, if you consider that the Circumference (whose Diameter is DE) which is the Locus of all the sought Points M, must cut the Line AB in two Points D, E, such that $AD : DB :: a : b$, and $AE : EB :: a : b$; because the Point M coinciding with D, the right Line AM does become AD; and BM, BD; and moreover, when the Point M coincides with E, the right Line AM does become AE, and BM, BE. For if the Line AB, produced on that Side (as is necessary) be divided in the Points D, E, so that $AD : DB :: a : b$, and $AE : EB :: a : b$; then it is evident, that the Line DE in both Cases will be the Diameter of the Circumference, being the Locus sought.

Case 3. Because a is $= b$ in this Case, the aforesaid Equation will be changed into this, $x = \frac{1}{2}c$; from whence if AP be taken equal to FIG. 189. $\frac{1}{2}AB$, and the right Line PM be drawn perpendicular on AB; the Line PM both ways indefinitely produced, will be * the Locus of all *Art. 307. the sought Points M; as is otherwise manifest by the Elements of Geometry.

EXAMPLE IV.

351. IF there be two Right Lines DE, DN, (indefinitely produced FIG. 190. both ways from the Point D) given in position on a Plane, together with the Point C without these Lines; and if a given Angle as CEM be supposed to move along so that its Vertex E be always in the Line DE, and the Side EC thereof (meeting DN in N) passes always through the Point C, and its other Side EM be always a third proportional to NC, CE; It is required to find the Locus made by all the Points M in that Motion.

Draw CA parallel to DN; and CB making an Angle at the Point B with DE, equal to the given Angle CEM, on that Side as is necessary, so that when CE coincides with CB, EM likewise coincides with DE. Now this Problem may be distinguish'd into three Cases: For either the Vertex (E) of the given Angle CEM moves along the right Line DE on the other Side of the Point B with respect to the

Point A; or between the Points B, A; or finally on the other Side of the Point A, with respect to the Point B.

Case 1. When the Vertex E moves along the Line DE on the other Side of the Point B with respect to A; draw the Line AQ towards the Point C, making the Angle BAQ with DE, equal to the Angle ABC; and through one of the Points sought, as M (which must be supposed as given) draw the Line MP parallel to AQ, meeting DE in P; then we shall have two similar Triangles CBE, EPM, for the Angles CBE, EPM, are each equal to the given Angle CEM; and moreover the Angles BCE, PEM, are also equal to one another; because in the Triangle CBE, the external Angle CEP or $CEM + PEM$, is equal to the two internal and opposite Angles BCE, CBE or CEM. Now if you call the given Lines AD, a; AB, b; BC, c; and the unknown and indeterminate ones AP, x; PM, y; AE, z; then according to the Conditions of the Problem, and because DN, AC, are parallel, we shall have these Proportions, $AD\,(a) : AE\,(z) :: CN : CE :: CE : EM :: CB\,(c) : EP\,(x-z) :: BE\,(z-b) : PM\,(y)$; from whence (by multiplying the Extremes and Means) we get these two Equations, *viz.* $ax - az = cz$ and $ay = zz - bz$, which, taking $f = a + c$ (for brevities sake), and getting out z, will be brought to this here $xx - \frac{bf}{a}x - \frac{ff}{a}y = 0$, containing only the two unknown Quantities x and y, with their Coefficients, and the Locus of this Equation, which is the Locus sought, may be * constructed after the following manner.

* Art. 310.

In AP assume the right Line $AF = \frac{bf}{2a}$ on the same Side AP as PM, and draw FL parallel to PM, and in the same assume $FG = \frac{bb}{4a}$, on the contrary Side AP with regard to PM. Then with the Diameter GL, whose Origin is the Point G, Parameter the Line $GH = \frac{ff}{a}$, and Ordinates, are parallel to AP, describe a Parabola tending the same way as PM. I say the indefinite Portion (OM) thereof contained in the Angle PAQ, will be the Locus of all the sought Points M.

For if the Line MQ be drawn from any Point M of the same parallel to AP, and meeting the Diameter GL in L; then we shall have ML or $PF = x - \frac{bf}{2a}$, and $GL = y + \frac{bb}{4a}$, and by the Nature of the Parabola $\overline{ML}^2\,(xx - \frac{bf}{a}x + \frac{bbff}{4aa}) = LG \times GH\ (\frac{ff}{a}y + \frac{bbff}{4aa})$

4 from

from whence by Transpofition there arifes the given Equation $xx - \frac{bf}{a}x - \frac{ff}{a}y = o$, which was requir'd to be conftructed.

Cafe 2. When the Vertex E moves along BA, in this Cafe it is manifeft, that the Points M fhall fall on the other Side of DE, becaufe the given Angle CEM will be always greater than the Angle CEP, which continually diminifhes; therefore $PM = -y$; and fince the fame Equation as above, is found by the like way of Argument as before; therefore the Portion (AGO) of the Parabola already defcrib'd fhall be the Locus of all the Points M, becaufe the abovefaid Equation fhall be had likewife by the Property of that Portion.

Cafe 3. When the Vertex moves on the other Side of the Point A, with refpect to the Points B; in this Cafe it is manifeft moreover, that all the fought Points M muft fall below the Line DE; and as in the firft Cafe, we fhall have $AD : AE :: CN : CE :: CE : EM :: CB : EP$. And therefore $AD : CB :: AE : EP$. From whence it appears, that EP is greater, lefs, or equal to AE, according as CB is greater, lefs, or equal to AD; and fo producing AQ below DE towards K, all the fought Points M, in the firft Cafe, do fall in the Angle BAK; in the fecond Cafe, in the Angle DAK the Complement of BAK; and laftly, in the third, on the right Line AK. Now fuppofe CB to be greater than AD; and becaufe if you make $PM = -y$ (fince the fame falls on the other Side of AP) there will not arife the fame Equation as in the firft Cafe; and fo the Conftruction of that Cafe will be ufelefs here; therefore calling, as before, AP, x; PM, y; and we fhall get this Equation, *viz.* $xx + \frac{bg}{a}x - \frac{gg}{a}y = o$, (wherein $g = c - a$) whofe Locus being that requir'd, will be the indefinite Portion (AM) of a Parabola different from the former one, and tending the oppofite way to that; and the fame may be conftructed * after the following manner. *Art. 310.

In AP affume $AS = \frac{bg}{2a}$ from A the contrary way to PM, and draw $ST = \frac{bb}{4a}$ parallel to AQ; and on the contrary Side, AP with regard to PM; then with the Diameter TS, whofe Origin is the Point T, Parameter a Line $= \frac{gg}{a}$, and whofe Ordinates are parallel to AP, defcribe a Parabola tending from P towards M. And the indefinite Portion (AM) thereof contain'd in the Angle PAK, fhall be the Locus of all the Points M in this laft Cafe, wherein CB is fuppos'd to be greater than AD. Therefore it is evident, that the fought Locus of all the Points M, confifts of two indefinite Portions of different Parabola's; one of which, as $AGOM$, tends towards C, and the other AM the contrary way, and both of them proceed from the Point A; for when CB

 the

the Side of the given Angle CEM falls in CA, which is parallel to DN; then it is manifest, that CN will become infinite, and so EM is $= o$, since we have always $NC : CE :: CE : EM$; that is, the Point M does coincide with the Points E and D: Therefore it appears, that AF is an Ordinate to the Diameter FG, and AS to the Diameter ST; and from hence arises the following general Construction.

In the indefinite right Line AP assume BO, BR, on both Sides the Point B, each equal to a fourth proportional to the three Lines DA, AB, BC; and through F, S, the middle Points of AO, AR, draw the Right Lines FG, ST, parallel to AQ, and each equal to a third Proportion to $4AD$, and AB, *viz.* FG on the contrary Side of AP, with regard to the Point C, and ST on the same Side. This being done, describe two Parabola's; one with GF, as a Diameter, and FA an Ordinate thereto; and the other with TS as a Diameter, and SA an Ordinate to the same: I say the indefinite Portions $MAGOM$ of those Parabola's, will be the compleat Locus of all the sought Points M.

For BO or $BR = \frac{bc}{a}$, and therefore AF or $\frac{1}{2}AO = \frac{1}{2}b + \frac{bc}{2a} = \frac{bf}{2a}$; and also AS or $\frac{1}{2}AR = \frac{bc}{2a} - \frac{1}{2}b = \frac{bg}{2a}$. Therefore, &c.

Here it may be observed (by the by) that if the given Angle whose Vertex moves along the Line DE, should be equal to the Complement of the Angle CEM to two Right Angles, every thing else remaining the same; that is, if the Points M should fall on the Line EM produced on the other Side of the Point E: Then the Locus of all the Points M would be the remaining Portions of the two Parabola's above describ'd.

If the Points A, B, C, should have a different Situation from what they are supposed to have in the Figure; two Equations would still be had, only differing from the former ones in some Signs; and so their Loci will consequently be Portions of Parabola's, that may be describ'd with the same ease, as before.

This Problem was proposed in the Journal of *Parma*, for *April*, in the Year, 1693. by *Count Vintimille*, which gave Occasion for Father *Saquerius* to publish a small Tract at *Milan*, wherein he acknowledges that he cou'd not solve the Problem, though by the Solutions of others it sufficiently appears that he is a good Geometrician.

EXAMPLE V.

FIG. 191. 352. AN indefinite Right Line AP being given in Position, together with two fixed Points A, C, one being in that Line, and the other without the same; let there be describ'd a Parabola AM, with

with AP as an Axis, whose Origin is in A, and any Line whatsoever for the Parameter; and from the given Point C let CM be drawn perpendicular to the Parabola. Now it is requir'd to find the Locus of all the Points M, whereof it is manifest that there are an infinite Number; because, the Parameters being indeterminate, there may be described an infinite Number of different Parabola's, to the same Axis AP, whose Origin is always in A.

Through the given Point C draw CB perpendicular to AP, and through one of the sought Points, as M, (supposed to be given) draw the Right Lines MP, MK, parallel to BC, AP, and the Tangent MT; then call the given Lines AB, a; BC, b; and the unknown and indeterminate ones AP, x; PM, y; from whence we have $CK = b - y$, and $MK = a + x$. But according to the Conditions of the Problem, the Angle CMT is a Right Angle: and consequently the Right-angled Triangles TPM, CKM, shall be similar; for if the same Angle KMT be taken from the Right Angles CMT, KMP, there will remain CMK, TMP, equal to one another: Therefore * $TP(2x) : PM(y) :: CK(b-y) : KM(a+x)$ and so (multiplying the Means and Extrems) we shall have this Equation $yy - by + 2xx + 2ax = 0$, whose Locus being that sought, will be * an Ellipsis, and may be * constructed after the following Manner. * Art. 22. and 23. *Art. 322. *Art. 324.

Draw $AD = \frac{1}{2}b$ perpendicular to AP, and on the same Side as PM, and draw the indefinite Right Line DL, parallel to AP, and in the same assume $DE = \frac{1}{2}a$ from D the contrary way to PM; on both Sides the Point E, take EF, EG, each equal to $\sqrt{\frac{1}{4}aa + \frac{1}{8}bb}$. Then describe an Ellipsis with the Axis FG, whose Parameter is a Line GH, being double to FG. I say the Portion of this Ellipsis AMO contained in the Angle PAD, is the Locus of the aforesaid Equation; and consequently of all the sought Points M, when they fall in that Angle.

For producing PM (if necessary) until it meets the Axis FG in L, we shall have the Ordinate $ML = \frac{1}{2}b - y$, and $EL = \frac{1}{2}a + x$, and by the Property of the Ellipsis, $FL \times LG$ or $\overline{EF}^2 - \overline{EL}^2$ $(\frac{1}{8}bb - ax - xx) : \overline{LM}^2(\frac{1}{4}bb - by + yy) :: FG : GH :: 1 : 2$; from whence by multiplying the Means and Extrems, and there arises $\frac{1}{4}bb - 2ax - 2xx = \frac{1}{4}bb - by + yy$. Therefore, &c.

Now if the Point M falls in the Angles BAD, BAR, you will find always the same Equation as above, from the Conditions of the Problem, and the Nature of the Ellipsis; observing to make $AP = -x$, and $PM = -y$, when the Point P falls on the other Side of the Origin A, and PM, on the other Side of the Line AP. From whence it follows, that the Portions of the Ellipsis (described as above) contain'd

tain'd in the Angles BAD, BAR, are the Loci of the Points M.

Here it must be observed, that not one of the sought Points M does fall in the Angle PAR, opposite to the Angle BAD, wherein the given Point C is situate. For if Right Lines as MP, MT, be drawn from any Point, taken in the said Angle PAR, perpendicular to AP, CM, then it is plain that the Points P, T, shall fall on the same Side the Point A; and consequently that Line cannot be a Tangent in M, as the Question requires.

If $AP(x)$ be supposed $= o$, then the foregoing Equation, *viz.* $yy - by + 2xx + ax = o$ will be changed into this $yy - by = o$, whose two Roots are $y = o$, and $y = b$: From whence it may be gathered, in drawing AO parallel and equal to BC, that the Locus of the sought Points M, shall pass through the Points A and O. After the same manner, if the Point P be supposed to fall on the other Side of the Origin A, and you make $AP(-x) = AB(a)$, then the same Locus shall pass through the Points B, C; so that the Ellipsis must be describ'd about the Rectangle $ABCO$: From whence arises the following new Construction.

*Art. 176. Form the Rectangle $ABCO$, and about * the same describe an Ellipsis, whose Axis FG being parallel to the Sides AB, OC, let be to its Parameter GH, as 1 to 2. Then it is manifest that the Ellipsis will be the Locus sought.

Scholium I.

353. If the Nature of any Curve as AM be expressed by the general *Art. 229. Equation $y^n = x^m a^{n-m}$ for * Parabola's of all Degrees (where the Letters m, n, denote the Index's of the Powers of y and x); then *Art. 237. we shall have $TP^*\left(\frac{n}{m}x\right) : PM(y) :: CK(b-y) : KM(a+x)$: and therefore $yy - by + \frac{n}{m}xx + \frac{n}{m}ax = o$, and the Locus thereof beiug that sought, will be an Ellipsis, which may be drawn according to the 322d, or else the 176th Articles, if you observe that the said Ellipsis must circumscribe the given Rectangle $ABCO$, and that its Axis FG being parallel to the Sides AB, OC, be to the Parameter GH in the given Ratio of m to n.

Scholium II.

Fig. 191. 354. If E the Centre of the Ellipsis should fall on the Origin (A) of the common Axis (AP) of all the Parabola's AM; and the Axis FG of the Ellipsis, in the Axis AP of the Parabola's: Then that Ellipsis would cut all the different Parabola's at right Angles. This Theorem may be expressed after the following Manner.

If

If there be an infinite Number of Parabola's as AM, of any Degree whatsoever, and the Line AP, whose Origin is always at the same Point A, be the common Axis of them all; and if there be an Ellipsis, whose Centre is the Point A, and whose Axis FG situate in AP, is to the Parameter thereof, as m the Index of the Power of $AP\,(x)$ to n, the Index of the Power of $PM\,(y)$ in the general Equation $y^n = x^m a^{n-m}$ expressing the Nature of the Parabola's AM: I say that Ellipsis shall cut all the Parabola's at right Angles. FIG. 192.

Through the Point M, in which the Ellipsis cuts any one of the Parabola's, draw the Line MT to that Parabola, and MS perpendicular to that Tangent. Then we are to prove, that MS touches the Ellipsis in the Point M. For doing of which, draw MP perpendicular to the Axis, and calling the indeterminate Quantities AP, x; PM, y; and the given Quantity FG, $2t$; then by the Property of the Ellipsis we shall have this Proportion $FP \times PG\,(tt - xx) : \overline{PM}^2\,(yy) :: m : n$. And therefore $myy = ntt - nxx$. But because TPM, TMS, are right Angles, therefore * $TP\,\left(\frac{n}{m}x\right) : PM\,(y) :: PM\,(y) : PS = \frac{myy}{nx}$, *Art. 237.

and consequently AS or $AP + PS = \frac{nxx + myy}{nx} = \frac{tt}{x}$ by putting $ntt - nxx$ for myy. Whence it appears, that $AP : AF :: AE : AS$, and so * the Line MS does touch the Ellipsis in the Point M. *W. W. D.* *Art. 51.

EXAMPLE VI.

355. LET there be suppos'd an infinite Number of Hyperbola's, which have all the same right Lines AP, AO, given in Position, and being at right Angles to one another, for their Asymptotes; and conceive an infinite Number of Perpendiculars, as CM, to be drawn from a given Point C to the said Hyperbola's. It is requir'd to find the Locus of all the Points M, wherein every of the right Lines CM meets the Hyperbola to which it is perpendicular. FIG. 193.

Draw the same Lines as in the last Example, and calling them by the same Names, we shall get * this Proportion, *viz.* $TP\,(x) : PM\,(y)$ $:: CK\,(b - y) : KM\,(a - x)$; from whence arises $yy - bx - xx + ax = 0$, whose Locus may be * thus constructed. *Art. 107. *Art. 330, or 335.

In the Asymptote AO parallel to PM, assume $AD = \frac{1}{2}b$, and draw DL parallel to AP; in which assume $DE = \frac{1}{2}a$ from D towards PM, and on both Sides the Point E, the Parts EF, EG, each equal to $\sqrt{\frac{1}{4}aa - \frac{1}{4}bb}$, or $\sqrt{\frac{1}{4}bb - \frac{1}{4}aa}$, according as a is greater or less than b. This being done, describe two opposite equilateral Hyperbola's to the Line FG, as a first Axis in the former Case, and as a second in the latter. I say, the Portions of these Hyperbola's contain'd

tain'd in the Angle PAO, will be the Locus of the abovesaid Equation, and consequently the Locus of all the sought Points M.

For producing PM (if necessary) to meet the Axis FG in L; then
* Art. 127. the Ordinate ML will be $= \frac{1}{2}b - y$, and $EL = x - \frac{1}{2}a$; and by * the Property of the Equilateral Hyperbola's $\overline{EL}^2 \pm \overline{EF}^2 (xx - ax + \frac{1}{4}bb) = \overline{LM}^2 (\frac{1}{4}bb - by + yy)$. Therefore, &c.

If a be $= b$, the aforesaid Construction will be impossible, because the Value of the Semi-axis EF or EG will become $= o$. And because the abovesaid Equation will become this $yy - ay - xx + ax = o$, or $yy - ay + \frac{1}{4}aa = xx - ax + \frac{1}{4}aa$; therefore extracting the square Root of both Sides, and there arises $y - \frac{1}{2}a = x - \frac{1}{2}a$, or $y = x$,
Fig. 194. and $\frac{1}{2}a - y = x - \frac{1}{2}a$, or $y = a - x$: And so if the Rectangle $ABCO$ be compleated, and the Diagonals AC, BO, be drawn, these Diagonals will be the Loci of all the sought Points M; for the Diagonal AC, is the Locus of the first Equation $y = x$, and the other Diagonal BO the Locus of the second $y = a - x$.

SCHOLIUM I.

Fig. 193. 356. IF the Nature of the Hyperbola's, whose Asymptotes are the right
* Art. 229. Lines AB, AO, be express'd by the general * Equation $x^m y^n = a^{m+n}$,
* Art. 237. then we shall have * $TP\left(\frac{n}{m}x\right) : PM\,(y) :: CK\,(b-y) : KM\,(a-x)$ and therefore $yy - by - \frac{n}{m}xx + \frac{n}{m}ax = o$, and the Locus of this
* Art. 330. Equation may be * thus constructed.

Find the Point E, as in the Example, and in DL both ways from E assume EF, EG, each equal to $\sqrt{\frac{1}{4}aa - \frac{m}{4n}bb}$, or $\sqrt{\frac{m}{4n}bb - \frac{1}{4}aa}$; according as naa is greater or less than mbb. Then describe two opposite Sections with the Line FG, as a first Axis in the former Case, and as a second in the latter, that may be to its Parameter in the given Ratio of M to N; and the Portions of these Sections contain'd in the Angle OAB, will be the Locus sought.

If $a : b :: \sqrt{m} : \sqrt{n}$, then the Equation $yy - by - \frac{n}{m}xx + \frac{n}{m}ax = o$, will be changed into this $yy - ay\sqrt{\frac{n}{m}} - \frac{n}{m}xx + \frac{n}{m}ax = o$,
Fig. 194. or $yy - ay\sqrt{\frac{n}{m}} + \frac{naa}{4m} = \frac{n}{m}xx - \frac{n}{m}ax + \frac{naa}{4m}$, and extracting the square Root of both Sides, and there comes out $y - \frac{1}{2}a\sqrt{\frac{n}{m}} = x\sqrt{\frac{n}{m}}$

$\frac{n}{m} - \frac{1}{2} a \sqrt{\frac{n}{m}}$, or $y = x \sqrt{\frac{n}{m}}$; and $\frac{1}{2} a \sqrt{\frac{n}{m}} - y = x \sqrt{\frac{n}{m}} - \frac{1}{2} a \sqrt{\frac{n}{m}}$, or $y = a \sqrt{\frac{n}{m}} - x \sqrt{\frac{n}{m}}$. Whence, if the Rectangle *AECO* be compleated, and the Diagonals *BO*, *AC*, are drawn; theſe Diagonals will be the Loci of all the ſought Points *M*. For the Diagonal *AC* is the Locus of the firſt Equation $y = x \sqrt{\frac{n}{m}}$, and the other Diagonal *BO* the Locus of the ſecond $y = a \sqrt{\frac{a}{m}} - x \sqrt{\frac{n}{m}}$.

• After the ſame manner, as in the Ellipſis, we prove, that the oppoſite Sections ſought muſt be deſcrib'd about the given Rectangle *ABCO*; and becauſe the Axis *FG*, parallel to the Sides *AE*, *OC*, is to its Parameter in the given Ratio of *m* to *n*; therefore you may deſcribe the oppoſite Sections (if you pleaſe) by means of the 176th Article. FIG. 193.

SCHOLIUM II.

357. If the Centre *E* of the Hyperbola *BFC* ſhould fall on the Point *A*, and its Axis *FG* in the Line *AP*; then that Hyperbola would cut all thoſe whoſe Aſymptotes are *AP*, *AO*, at right Angles; which may be thus expreſs'd. FIG. 193.

If there be an infinite Number of Hyperbola's of any Degree whatſoever, having the ſame ſtrait Lines *AP*, *AO*, at right Angles to one another, for their Aſymptotes; and if there be a common Hyperbola *FM*, whoſe Centre is *A*, and whoſe firſt Axis *FG* ſituate in *AP*, be to its Parameter as *m* the Index of the Power of *AP* (x) to *n*, the Index of the Power of *PM* (y) in the general Equation $x^m y^n = a^{m+n}$, which expreſſes the Nature of the Hyperbola's *MAM*. I ſay, the Hyperbola *FM* does cut all thoſe different Hyperbola's at right Angles. FIG. 195.

Through the Point *M*, wherein the Hyperbola *FM* cuts any one of the different ones, as *MAM*, draw the Tangent *MT* to the ſame; and the right Line *MS* perpendicular to that Tangent: Now we are to prove, that the Angle *TMS* is a right Angle. To do this, draw *MP* perpendicular to the Aſymptote *AP*; and calling the indeterminate Quantities *AP* (x); *PM*, y; and the given Quantity *FG*, $2t$; then by the Property of the Hyperbola *FM*, we have this Proportion, *viz.* $FP \times PG\ (xx - tt) : \overline{PM}^2\ (yy) :: m : n$, and therefore $myy = nxx - ntt$. But ſince the Angles *TPM*, *TMS*, are right Angles, there ariſes TP* $\left(\frac{n}{m} x\right) : PM\ (y) :: PM\ (y) : PS = \frac{myy}{nx}$. And conſequently AS or $AP - PS = \frac{nxx - myy}{nx} = \frac{tt}{x}$, by ſubſtituting $nxx - ntt$ for *Art. 237.

myy. From whence it appears, that AS is a third Proportional to
* Art. 121. AP, AF; and so * the Line MS does touch the Hyperbola FM in the Point M. W. W. D.

EXAMPLE VII.

FIG. 196. 358. THE Parabola BAC being given, it is required to find the Locus of all the Points M, being such, that two Tangents MB, MC, to the Parabola being drawn from them; the Angle BMC contained by these Tangents, may be equal always to a given Angle.

Here there are three different Cases according as the given Angle BMC is acute, obtuse, or right.

* Art. 160. *Case* 1. When BMC is an acute Angle, draw * AD the Axis of the given Parabola BAC, meeting the Tangents MB, MC, in the Points F, G, also from the Points of Contact B, C, and the Point of Concurrence M draw the Lines BD, CE, MP, perpendicular to the Axis. This being done, draw MN making the Angle FNM with the Axis AD equal to FMG the Complement of the given Angle BMC, to two right Angles, and call the unknown Quantities AP, x; PM, y; AF, s; AG, t; and the Parameter of the Axis AD, *viz.* AV, a; which is given, because the Parabola is given. Now because FPM is a Right-angl'd Triangle, therefore the Square $\overline{FM}^2 = ss - 2sx + xx + yy$, which being divivded by FG $(s-t)$ will give $\frac{ss-2sx+xx+yy}{s-t} = FN$, because the Triangles FGM, FMN, are similar; and therefore PN or $FP - FN = \frac{sx+tx-st-xx-yy}{s-t}$. Now by means of the given Parabola BAC, seek the Values of $s+t$, st, and $s-t$ with respect to x and y, that so they being substituted in the Value of PN, this Line may be expressed only by x and y. Which must be done thus.

The similar Triangles FPM, FDB; and GPM, GEC, do give
* Art. 22. FP $(s-x)$: PM (y) :: FD * $(2s)$: BD * $(\sqrt{as})$. And GP $(x-t)$:
* Art. 7. PM (y) :: GE $(2t)$: CE $(\sqrt{at})$. From whence may be form'd the following Equations, *viz.* $ss - 2xs - \frac{4yy}{a}s + xx = o$, and $tt - 2xt - \frac{4yy}{a}t + xx = o$; that is, (making $p = 2x + \frac{4yy}{a}$ t for brevities sake) $ss - ps + xx = o$, and $tt - pt + xx = o$; then substract the second Equation from the first, and you will get $ss - tt - ps + pt = o$, which being divided by $s-t$, and then $s+t$ is $= p$, and so $s = p - t$, and $ss = ps - ts = ps - xx$, by the first Equation, therefore $st = xx$. Lastly, if $4xx$ the Value of $4st$ be taken from pp, the Value of

ss -

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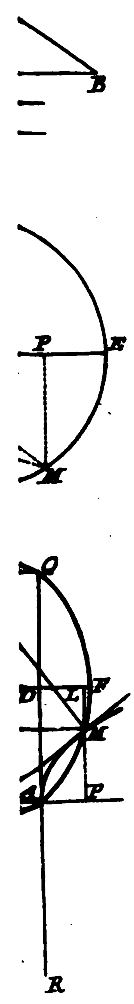


$s + 2ts + tt$, there will arise this Equation, *viz.* $ss - 2st + tt = pp - xx$, and extracting the Square Root of both Sides, we have $s - t = \sqrt{pp - 4xx} = \frac{4y\sqrt{ax+yy}}{a}$, substituting $2x + \frac{4yy}{a}$ for p.

Now if for $s+t$, st, and $s-t$, you substitute their Values $2x + \frac{yy}{a}$, xx, and $\frac{4y\sqrt{ax+yy}}{a}$, in $\frac{sx+tx-st-xx-yy}{s-t}$ and then PN will be $= \frac{4xy-ay}{4\sqrt{ax+yy}}$. Now if NQ be taken in the Axis equal to its Parameter AV (a), and QT be drawn parallel to PM, meeting the right Line MN (produced according to Necessity) in the Point T; then it is plain that the Line QT will be given, because in the Right-angl'd Triangle NQT, the Angle QNT, which is equal to the given Angle BMC is given; and moreover the Side NQ being equal to the Parameter AV to the Axis. Now let the given Line QT be $= b$, then because the Triangles NPM, NQT, are similar, therefore NP $\left(\frac{4xy-ay}{4\sqrt{yy+ax}}\right) : PM\,(y) :: a : b$, and so $4a\sqrt{yy+ax} = 4bx - ab$, and clearing the Equation from Surds, and we have $yy - \frac{bb}{aa}xx + ax + \frac{bb}{2a}x - \frac{1}{16}bb = 0$, whose Locus (being that sought) may be * thus constructed. *Art. 330, and 332.

In AD, the Axis of the Parabola, assume $AH = \frac{1}{2}a + \frac{a^3}{2bb}$ from A towards PM, and on both Sides the Point H take HI, HK, each equal to $\frac{aa\sqrt{aa+bb}}{2bb}$; and describe an Hyperbola KM with IK as a first Axis, having the same Proportion to the Parameter KL as aa to bb; then I say, that Hyperbola will be the Locus of the Equation found as above.

For $HP = x - \frac{1}{2}a - \frac{a^3}{2bb}$, and by the Property of the Hyperbola $\overline{HP}^2 - \overline{HK}^2$ $(xx - \frac{1}{2}ax - \frac{a^3}{bb}x + \frac{1}{16}aa) : \overline{PM}^2\,(yy) :: IK : KL :: aa : bb$. From whence may be gotten the abovesaid Equation, by multiplying the Means and Extremes.

Here it is necessary to observe, that FN shall be always less than FP; because the Angle FNM, which was taken equal to the Complement of the given Angle, is obtuse. Therefore $\frac{4xy-ay}{4\sqrt{yy+ax}}$ the Value of $FP - FN$ must be positive; and consequently x must always exceed

ceed $\frac{1}{4}a$. From whence it appears, that although there be a Portion of the Hyperbola opposite to KM, which is contain'd in the Angle PAV made by the Line AP, and the right Line AV drawn parallel to PM towards the same Parts, yet that Portion cannot be the Locus of the Points M; because AI being less than $\frac{1}{4}a$, the variable Quantity AP, which then shall be less than AI, will be much more less than $\frac{1}{4}a$.

Case 2. When the given Angle is obtuse. Supposing that the Points M do fall in the Angle PAV, and reasoning after the same manner as in the first Case, you will get the same Equation as there, and consequently the Construction of the Locus shall be the same also. But here it may be observ'd, that FN shall be greater than FP, and so the Value $\frac{4xy-ay}{4\sqrt{yy+ax}}$ of $FP-FN$ will become negative; from whence it follows, that x shall be always less than $\frac{1}{4}a$, and therefore the Locus sought will be the Portion of the Hyperbola tending the same way as the Parabola, being contain'd in the Angle PAV. And because the same Equation (as in *Case* 1.) arises, supposing the Points M to fall in the Angle DAV, therefore that whole Hyperbola shall be the Locus of all the Points (M) sought.

From hence it is manifest, that if an Hyperbola, as KM, be the Locus of all the Points M, when the given Angle BCM is acute; then the opposite Hyperbola shall be the Locus of all those Points, when the given Angle shall be equal to the Complement of the Angle BMC to two right ones, because then the given Lines a and b, which determine the Construction of the Hyperbola's, will remain the same.

FIG. 196, 197. *Case* 3. When the given Angle is a right Angle. Here it is manifest, that FN is equal to FP, and so $\frac{4xy-ay}{4\sqrt{yy+ay}}$ the Value of $FP-FN$ shall be $=0$. Whence if $AP=\frac{1}{4}a$ be taken in the Axis AD produced towards its Origin A, and the indefinite right Line PM be drawn perpendicular to the same; then it is * manifest, that PM, which is the Directrix of the Parabola, (as appears from the Definitions in Book 1.) shall be the Locus sought.

* Art. 306.

COROLLARY.

FIG. 196, 197. 359. IF the Semi-second Axis HO be drawn, as also the Hypothenuse KO; then the right-angled Triangles KHO, NQT, shall be similar: For because the second Axis is a mean Proportional between the first Axis IK, and its Parameter KL, therefore $\overline{KH}^2 : \overline{HO}^2 :: IK : KL :: aa : bb$, and so $KH : HO :: NQ\,(a) : QT\,(b)$. Therefore the Angle HKQ (which by *Def.* 11. *Book* 3. is equal to the half of the Angle

Angle form'd by the Asymptotes of the Hyperbola KM) shall be equal to the Angle QNT, that is, to the given Angle BMC; and we shall have $NQ(a) : QT(b) :: KH\left(\frac{aa\sqrt{aa+bb}}{2bb}\right) : HO = \frac{a\sqrt{aa+bb}}{2b}$;

and $NQ(a) : NT(\sqrt{aa+bb}) :: KH\left(\frac{aa\sqrt{aa+bb}}{2bb}\right) : KO = \frac{a^3+abb}{2bb}$.

Now if the Hypothenuse (KO) of the right-angled Triangle (KHO) made by the Semi-axes HK, HO, be laid from the Centre H in the first Axis IK to R, and S; then it is manifest, * that R and S, shall be the two Foci of the Hyperbola KM, and that opposite to it; and *Art. 74.

$RA = \frac{1}{4}a$, because $HR = \frac{a^3+abb}{2bb}$, and $AH = \frac{1}{4}a + \frac{a^3}{2bb}$. Whence the Focus R of the Hyperbola KM, is * also the Focus of the Parabola BAC; and $SR\left(\frac{a^3+abb}{bb}\right) : HO\left(\frac{a\sqrt{aa+bb}}{2b}\right) :: HO\left(\frac{a\sqrt{aa+bb}}{2b}\right)$ *Def. 3, 4, 5. I.

$: AR(\frac{1}{4}a)$, because if the Means and Extremes be multiply'd, the same Product will be found. From whence arises the following Theorem.

If RA be taken from R towards S in the focal Distance SR of an Hyperbola KM, equal to a third Proportional to that Distance SR and HO half of the second Axis; and if a Parabola BAC be describ'd *, having the Point R for the Focus, and the Line AR whose Origin is A, for the Axis; and if from any Point M of the Hyperbola KM, there be drawn two Tangents MB, MC, to that Parabola. I say, the Angle BMC contain'd by these Tangents, shall be always equal to $\frac{1}{2}$ the Angle form'd by the Asymptotes; and if the Point M be assum'd on the opposite Hyperbola, the Angle contain'd by the Tangents, will be equal always to the Complement of $\frac{1}{2}$ the Angle form'd by the Asymptotes to two right Angles. Fig. 196. * Art. 4.

EXAMPLE VIII.

360. AN indefinite right Line BAP being given in Position on a Plane, together with two fixed Points A, D, the one in that Line, and the other without the same: It is required to find the Locus of all the Points M, whose Property may be such, that two right Lines MA, MD, being drawn from any one of them, to the fixed Points A, D: The Line AM may be always equal to ME, that part of the other Line DM, taken between the Point M, and the Point E wherein it meets the Line BAP. Fig. 198.

Draw the right Lines BD, MP, from the given Point D, and the Point M (supposed to be one of the Points sought) perpendicular to AP, and call the given Lines AB, $2a$; BD, $2b$; and the

unknown and indeterminate ones AP, x; PM, y; then we shall have $AP = PE$, because (by the *Hyp.*) $AM = ME$. And because the Triangles EBD, EPM, are similar, therefore EB or $AE - AB$ $(2x - 2a) : BD\ (2b) :: EP\ (x) : PM\ (y)$. And multiplying the Extremes and Means, and we shall have this Equation $xy - ay = bx$, containing the Condition specified in the Problem, and the Locus thereof,
* Art. 337. which is * an Equilateral Hyperbola between its Asymytotes may be thus constructed.

Draw the Line AD which bisect in C, and through C draw the right Lines CF, CG, the one parallel, and the other perpendicular to AP: Then between the Asymptotes CF, CG, both ways indefi-
* Art. 130. nitely produced from the Point C * describe * two opposite Hyperbola's
131. DM, AM, (which * are equilateral) through the Points A, D.
* Def. 16. I say, these Hyperbola's shall be the compleat Loci of all the sought
III. Points M.

For the Asymptotes CF, CG, do divide the right Lines AB, BD, into two equal Parts in the Points L, K, because AD is bisected in C; and therefore when the Points P do fall on AB infinitely produced towards B, as is supposed in the Calculus, the Line PL or CH is $=$
* Art. [illegible] $x - a$, and $HM = y - b$; and by * the Property of the Hyperbola $CH \times HM\ (xy - ay - bx + ab) = CK \times KD\ (ab)$: That is, $xy - ay = bx$.

Now if the Points P be supposed to fall upon BA, produced infinitely towards A, or on the determinate Part AB, the same Equation $xy - ay = bx$ will always be had, from the Condition denoted in the Problem and the Nature of the Hyperbola AM, or DM, observing to make $AP = -x$, and $PM = -y$, when they fall on the other Side the Point A and the Line AP. Therefore, &c.

COROLLARY.

361. HENCE the Parts MR, MS, of the right Lines AM, DM, taken from the Point M to the Asymptotes, are equal to one another. For 1. when the Asymptote, as CF, is parallel to the Line AP, then the Angle RSM is equal to the Angle AEM, and the Angle SRM equal to the Angle MAE. 2. When the Asymptote, as CG, is perpendicular to AP, then the Angle RSM shall be the Complement of the Angle AEM to one right Angle, (since SLE is a right Angle Triangle) and also the Angle SRM or ARL vertically opposite to it is the Complement of the Angle EAM to a right Angle, because RAL is a Right-angl'd Triangle. Therefore because the Angles EAM, AEM, are equal, the Triangle RMS shall be an Isosceles Triangle, and so the Sides MR, MS, shall be equal to one another.

another. Now this Corollary furnishes us with the following Theorem.

If from any Point M of an equilateral Hyperbola there be drawn two right Lines MD, MA, to the Extremities of (AD) one of the first Diameters thereof; meeting the Asymptotes in the Points R, and S: I say the Parts MR, MS, of those Lines shall be equal between themselves.

EXAMPLE IX.

362. TWO Circles EGF, BNO, whose Centres are C and A, being given, and if through any Point G of the Circle EGF, there be drawn the indefinite Tangent GNO cutting the Circle BNO in two Points N, O, and if the Tangents NM, OM, be drawn through N and O; it is requir'd to find the Locus of all the Points of Concurrence M. FIG. 199.

Draw MP perpendicular to CA, which passes through C and A the Centres of the given Circles; also draw the right Lines CG, AM, which shall be parallel, because both of them is perpendicular to the right Line GO which meets them in the Points G and Q; and call the given Quantities AB or AO, a; CE, or CF, or CG, b; CA, c; and the unknown and variable Quantities AP, x; PM, y. Now because the right-angl'd Triangles AOM, AQO, are similar, therefore $AM(\sqrt{xx+yy}) : AO(a) :: AO(a) : AQ = \frac{aa}{\sqrt{xx+yy}}$. And if CH be drawn parallel to GO, meeting MA (produced according to Necessity) in the Point H; then (by the Similarity of the Right-angl'd Triangles MAP, CAH) $PA(x) : AM(\sqrt{xx+yy}) :: AH$ or $CG - AQ\left(b - \frac{aa}{\sqrt{xx+yy}}\right) : AC(c)$; and so $b\sqrt{xx+yy} = aa + cx$, that is (freeing the Equation from Surds) $yy + \frac{bb-cc}{bb}xx - \frac{2aac}{bb}x - \frac{a^4}{bb} = o$, and the Locus of this is * a Parabola, Ellipsis or Hyperbola, according as $CE(b)$ is equal, greater, or less, than $CA(c)$. The Construction of this last Case is as follows. * Art. 345.

In the Line AP assume $AR = \frac{aac}{cc-bb}$ from A the contrary way to P; and on both Sides the Point R take RI, RK, each equal to $\frac{aab}{cc-bb}$; and with IK as a first Axis, having $KL = \frac{2aa}{b}$ for its Parameter describe an Hyperbola. I say, the indefinite Portion DM of this Hyperbola,

perbola, which is contain'd in the Angle PAD made by the Line AP, and the right Line AD drawn parallel to PM on the fame Side AP, with regard to AP, fhall be the Locus of the aforefaid Equation.

For by the Property of the Hyperbola, $\overline{RP}^2 - \overline{RI}^2 \left(\frac{a^4 + 2aacx}{cc - bb} + xx\right) : \overline{PM}^2 (yy) :: IK \left(\frac{2aab}{cc-bb}\right) : KL \left(\frac{2aa}{b}\right)$; which gives again the aforefaid Equation.

Now if the Points M be fuppos'd to fall in the Angle KAD, adjoining to the Angle PAD, the aforefaid Equation will yet be found, in making $AP = -x$; therefore ID the determinate Portion of the Hyperbola IM, together with half of the whole Hyperbola oppofite to the fame, fhall be the Locus of all the Points M, and fo the two oppofite Hyperbola's do make up the compleat Locus of all the fought Points M: Where it muft be obferv'd, that the Portion SIT, contain'd in the Circle BNO, is ufelefs, becaufe not one of the Points of Concurrence of the two Tangents NO, OM, to that Circle can fall within the fame.

Here it is proper to obferve, that $RA \left(\frac{aac}{cc-bb}\right)$ is $= \sqrt{\overline{KR}^2 + \frac{1}{4} IK \times KL}$, as will appear by putting for thofe Lines their Analytick Values; and fo fince the Rectangle $IK \times KL$ is equal to
* Art. 74. the Square of $\frac{1}{2}$ the fecond Axis, the Point A fhall be * one of the Foci of the Hyperbola IM. And becaufe AI or $AR - RI$ is $= \frac{aac - aab}{cc - bb} = \frac{aa}{c+b}$, and $AK = AR + RK = \frac{aac + aab}{cc - bb} = \frac{aa}{c-b}$: Therefore the aforefaid Conftruction may be fhortened after this manner.

In the Line AC affume AD, AK (from A towards C) as third
* Art. 76. Proportionals to $AF (c+b)$, $AB (a)$, and to $AE (c-b)$, $AB (a)$; and with the firft Axis IK, and the Focus A defcribe * two oppofite Hyperbola's. Then it is evident that thofe Hyperbola's fhall be the Loci of all the fought Points M.

When $CE (b)$ is greater than $CA (c)$, the Conftruction of the Ellipfis, which is the Locus of all the fought Points M, will be after the fame manner as that for the Hyperbola, obferving to affume AK on the other Side the Point A with regard to C. And finally when $CE (b)$
Fig. 200. $= CA (c)$, then you need only affume AI in AC from A towards C, equal to a third Proportional to AF, and AB, and afterwards defcribe a Parabola, with the Point A as a Focus, and the Line AI, whofe Origin is in I, as the Axis.

COROLLARY I.

For the ELLIPSIS *and opposite* SECTIONS.

363. HENCE if any Circle BNO be describ'd about (A) one of the Foci of an Ellipsis or the opposite Sections, having IK, as a first Axis; and if AE, AF, be taken in that Axis, as third Proportionals to AK, AB, and to AI, AB, and the Circle EGF be describ'd with EF, as a Diameter: Then it is evident, if two Tangents MN, MO, be drawn from any Point M of the Section to the Circle BNO, that the Line ON joining the Points of Contact (being produced according as is necessary) shall always touch the Circle EGF. FIG. 199.

COROLLARY II.

For the PARABOLA.

364. HENC also if any Circle BNO be described from A the Focus of a Parabola IM, whose Axis is LA, and Origin I; and if AF be taken in that Axis from A towards its Origin, as a third Proportional to AI, AB, and a Circle AGF be described with the Diameter AF: And lastly, if from any Point M of the Parabola, there be drawn two Tangents MN, MO, to the Circle BNO: Then the Line NO joining the Points of Contact, (being produced according to Necessity) shall touch the Circle AGF in the Point G. FIG. 200.

EXAMPLE X.

365. AN indefinite right Line AP being given on a Plane, together with the fixed Point F without the same: It is requir'd to find the Locus of all the Points M being such, that a right Line MP drawn from any one of them perpendicular to AP, and another right Line MF being drawn to the Point F; the Ratio of MP to MF may be always the same, suppose as the given Quantity a to b. FIG. 201, 202, 203.

Draw the right Line FA from the given Point F, perpendicular to AP, and the Line MQ from the Point M (supposed to one of those sought) parallel to AP; and call the given Quantity AF, c, and the unknown and variable ones AP, x; PM, y; (these are at right Angles to one another.) Now because MQF is a Right-angl'd Triangle, therefore $\overline{MF}^2 = \overline{FQ}^2 (cc-2cy+yy) + \overline{MQ}^2 (xx)$; and according to the State of the Problem $\overline{MP}^2 (yy) : \overline{MF}^2$

&c.

$(cc - 2cy + yy + xx) :: aa : bb$; from whence (multiplying the Means and Extremes) there arises this Equation $aayy - bbyy - 2aacy + aaxx + aacc = 0$, the Construction of whose Locus, (wherein are three different Cases, according as a is greater, less, or equal to b,) is as follows.

Case 1. If the aforesaid Equation be divided by $aa - bb$ there

* *Art.* 324. will arise $yy - \frac{2aac}{aa-bb}y + \frac{aa}{aa-bb}xx + \frac{aacc}{aa-bb} = 0$, whose Locus is * an Ellipsis, and may be constructed thus.

Fig. 201. In AF assume $AC = \frac{aac}{aa-bb}$, from A towards F; and drawing KH through the Point C parallel to AP, assume in the same the Parts CH, CK, on both Sides the Point C, each equal to $\sqrt{\frac{bbcc}{aa-bb}}$. This being done with the Axis KH, having the same Proportion to its Parameter KL as $aa - bb$ to aa, describe an Ellipsis. I say, the same will be the Locus of the aforesaid Equation, and consequently of all the sought Points M.

For by the Property of the Ellipsis $KE \times EH$, or $\overline{CH}^2 - \overline{CE}^2$ $\left(\frac{bbcc}{aa-bb} - xx\right) : \overline{EM}^2$ $\left(\frac{a^4cc}{\overline{aa-bb}^2} - \frac{2aacy}{aa-bb} + yy\right) :: KH : KL :: aa - bb : aa$; from whence, multiplying the Means and Extremes, and there will arise the same Equation as above.

Because $\overline{CH}^2 : \overline{CB}^2 :: KH : KL :: aa - bb : aa$, therefore the Semi-axis CB or CD is $= \frac{abc}{aa-bb}$, and so DF or $DC + CF$ is $= \frac{abc+bbc}{aa-bb} = \frac{bc}{a-b}$, and FB or $CB - CF$ is $= \frac{abc - bbc}{aa-bb} = \frac{bc}{a+b}$. Therefore

* *Art.* 35. $DF \times FB$ is $= \frac{bbcc}{aa-bb} = \overline{CH}^2$; and consequently the Point F is * one of the Foci of the Ellipsis, and BD is the great Axis. Now from hence arises a much easier Construction than that foregoing, which is this.

In FA assume $FB = \frac{bc}{a+b}$ from F towards A, and $FD = \frac{bc}{a-b}$ the contrary way, also assume $DG = BE$ from D towards F; then

* *Art.* 36. describe * an Ellipsis with the Points F, G, as Foci, and the Axis BD, and it is evident that the same shall be the Locus sought.

Case 2. Hence we have $yy + \frac{2aac}{bb-aa}y - \frac{aa}{bb-aa}xx - \frac{aacc}{bb-aa} = 0$; because a is less than b. And the Locus of this Equation will be two opposite ions, which may be describ'd by the 332d Article. After having

made the ſame Obſervations as in the precedent Caſe, the following Conſtruction may be gathered.

In FA aſſume $FB = \frac{bc}{b+a}$, and $FD = \frac{bc}{b-a}$ from F towards A: FIG. 202.
Alſo take $DG = BF$ from D the contrary way to F; then with the Points F and G, as Foci, and BD, as a firſt Axis, deſcribe * two oppoſite Sections BM, DM; and the ſame ſhall be the Locus of all the ſought Points M. * *Art.* 76.

Caſe 3. Here the general Equation $aayy - bbyy - 2aacy + aaxx + aacc = 0$ will become this $xx - 2cy + cc = 0$, becauſe a is $= b$, and the Locus thereof is a Parabola, which may be conſtructed eaſily by Article the 310th. But it appears at once, without any manner of Calculus, if a Parabola be deſcrib'd with the Line AP as a Directrix, and the Point F as a Focus, (according to the Directions of Def. 1.) that the ſame will be the Locus requir'd. FIG. 223.

COROLLARY I.

366. IN the firſt Caſe, it is evident that $CF\left(\frac{bbc}{aa-bb}\right) : CB\left(\frac{abc}{aa-bb}\right) :: CB\left(\frac{abc}{aa-bb}\right) : CA\left(\frac{aac}{aa-bb}\right) :: a : b$; the ſame will be found alſo in the ſecond Caſe: From hence ariſes the following Theorem.

In an Ellipſis or the oppoſite Sections, whoſe Centre is the Point C, and two Foci the Points F and G, and firſt Axis the Line BD, if CA be taken equal to a third Proportional to CF, CB, from C towards the Focus F; and if the indefinite Line AP be drawn perpendicular to BD: Then if from any Point M of the Section, be drawn the right Line MP perpendicular to AP, and the right Line MF to the Focus F: I ſay the Ratio of MP to MF, ſhall be always the ſame as the Ratio of the firſt Axis BD to FG, the Diſtance between the Foci. FIG. 201, 202.

In the following Corollaries the indefinite right Line AP is called a *Directrix*, as well in regard to the Ellipſis and oppoſite Sections, as to the Parabola. From whence it appears to be eaſy to deſcribe a Conick Section, whoſe Focus is the given Point F, Directrix the right Line AP given in Poſition, that ſhall paſs through a given Point M; for drawing the Line MF to the Focus F, and MP perpendicular to the Directrix AP, and calling the given Quantities MP, a; MF, b; then you need only deſcribe the Locus of the Points M being ſuch, that MP be always to MF as a to b.

COROLLARY. II.

FIG. 204, 205. 367. IF any two Points M and N, of a Conick Section be joined by a right Line meeting the Directrix in C; and if the right Lines FM, FN, FC, be drawn from the Focus F: I say the Line FC does bisect NFH the Complement of the Angle NFM to two right Angles, when the Points M and N be taken on the Parabola, Ellipsis, or Hyperbola, and the Angle NFM, when they be taken on the opposite Sections.

For if the Perpendiculars MP, NQ, be drawn to the Directrix, and if the Line ND be drawn parallel to MF; then because the Triangles MPC, NQC, and MFC, NDC, are similar, therefore $MP : NQ :: MC : NC :: MF : ND$. And consequently $MP : MF :: NQ : ND$. But by the Property of the Conick Section, having PQ for the Directrix, and F for a Focus, we have $MP : MF :: NQ : NF$. Therefore the Lines ND, NF, shall be equal between themselves; that is, (in the first Case) the Angle NDF or CFM, shall be equal to the Angle CFN, and (in the second) the Angle FDN or CFH shall be equal to the Angle CFN. *W. W. D.*

COROLLARY III.

FIG. 204. 369. HENCE appears the manner of describing a Parabola, Ellipsis, or Hyperbola, which shall pass through three given Points M, N, O, and have the given Point F for a Focus.

Through the Focus F draw the right Lines FC, FE, which do bisect NFH, NFK, the Complements of the given Angles MFN, OFN; and through the Points C, E, wherein FC, FE meet the Lines MN, ON, (which join the given Points) draw the indefinite Line CE. Then if a Conick Section be described through the Point M, with the Line CE as the Directrix, and the Point F, as a Focus; it is manifest by the precedent Corollary, that that Conick Section will also pass through the two other Points N and O.

COROLLARY IV.

FIG. 205. 369. FROM the second Corollary arises a way of describing two opposite Hyperbola's, having the Point F as a Focus; so that one of them shall pass through two given Points M, O, and the other through one given Point N.

Through the Point F, draw the Line FE, bisecting HFO, the Complement of the Angle MFO (formed by the right Lines FM, FO, drawn from the Point F to the Points M, O, being both in the same Hyperbola.) Also thro' the same Point F, draw the Line FC;

FC, bisecting the Angle *MFN*, (form'd by the right Lines *FM*, *FN*, drawn from the Point *F* to the Points *M* and *N*, falling on the opposite Hyperbola's.) Through the Points *E*, *C*, wherein the Lines *FE*, *FC*, meet the right Lines *MO*, *MN*, (that do join the given Points) draw the indefinite Line *EC*. Lastly, describe two opposite Hyperbola's with the Point *F*, as a Focus, and the right Line *EC*, for the Directrix, so that one of them may pass through the Point *M*; and then it is evident, that these Hyperbola's will answer what is propos'd.

COROLLARY V.

370. THE same Things being premis'd as in the second Corollary; FIG. 204. it is manifest, that the Angle *MFN*, the Difference between *CFH* or *CFN* the Complement thereof to two right Angles, diminishes more and more, according as the Point *N* accedes to *M*, so that the same will vanish quite, when the Point *N* coincides with *M*; therefore the Angle *CFM* shall then be equal to its Complement to two right Angles, and consequently will be a right Angle. And because the Line *MD* does then become the Tangent *MT*, since * the same passes through two Points of the Curve infinitely near to each other; therefore, from hence we have a general and new way of drawing a Line (*MT*) to touch a Conick Section in the given Point *M*, the Focus, together with the Axis passing through that Focus, being given. *Art. 188.

For finding the Directrix according to the Directions in *Coroll.* 2, from the given Point *M* draw the right Line *MF* to the Focus *F*, and draw the right Line *FT* perpendicular to *MF*, meeting the Directrix in *T*; then if *MT* be drawn through the Point *T* and the given Point *M*, the same will touch the Section in *M*.

EXAMPLE XI.

371. TWO Angles *KAM*, *KBM*, moveable about the fixed Points FIG. 206. *A*, *B*, being given upon a Plane, together with an indefinite right Line *FK*, not passing through those Points; let the Point of Concurrence (*K*) of the two Sides *AK*, *BK*, move along the right Line *FK*: Now it is requir'd to find the Nature of the Curve describ'd by the Intersection (*M*) of the other two Sides *AM*, *BM*, produced, when necessary, on the other Side the Points *A* and *B*.

Upon *AB*, as a Chord, describe the Segment of a Circle, on the other Side the Point *M*, containing an Angle *BDA* equal to four right Angles, minus the two given Angles *KAM*, *KBM*; and compleating the whole Circle whereof *BDA* is the Segment, it may happen that the indefinite right Line *FK* does fall quite without that

 Circle,

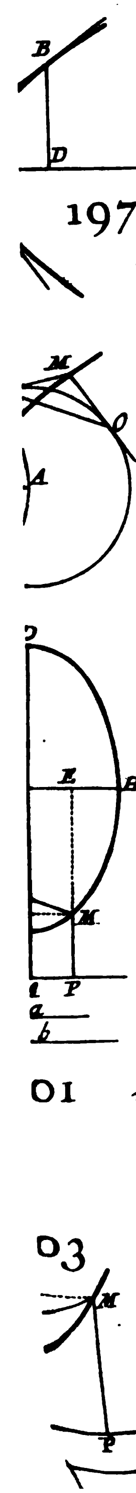
Circle, within the same, or finally touches it: And so there are three Cases, which I shall explain in particular.

Case 1. From C the Centre of the Circle $BDAE$, draw CF perpendicular to FK meeting the Circle in the Points D, E; and through the Point D (being nigher to the Line FK than the Point E) let the two Sides DA, DB, of the two Angles DAP, DBQ, equal to the two Angles KAM, KBM, pass, which Sides being produced towards D, let meet the Line FK in the Points G and H. And by Construction, the Angle BDA plus the two Angles DAP, DBQ, are equal to four right Angles; and since the same Angle BDA plus the two Angles DAB, DBA, are equal to two right Angles; therefore the Angles BAP, ABQ, are equal to two right ones; and so the Lines AP, BQ, are parallel between themselves. This being laid down.

From the Point K draw KR, KS, perpendicular to the Sides AD, BD; and from the Points A, M, the Lines AI, MP, perpendicular to the two other Sides BQ, AP, which meet BQ, in the Points I and Q. Now let the given Quantities FE be $=a$, $FD=b$, $BI=c$, $AI=d$, $FG=g$, $FH=h$, $DG=m$, $DH=n$: and the unknown Quantities $FK=z$, $AP=x$, $PM=y$; then because GDF, GKR, are Right-angl'd similar Triangle: Therefore $GD\,(m):GF\,(g)::GK\,(z-g):GR=\frac{gz-gg}{m}$. And $GD\,(m)\,DF::(b)::GK\,(z-g)\,KR=\frac{bz-bg}{m}$. But because GDF, EDA are also Right-angl'd similar Triangles, therefore $GD\,(m):DF\,(b)::ED\,(a-b):AD=\frac{ab-bb}{m}$; and consequently $AD+DG$ or AG is $=\frac{ab-bb+mm}{m}$, and $AG+GR$ or $AR=\frac{ab-bb+mm+gz-gg}{m}=\frac{ab+gz}{m}$; since DFG being a right-angl'd Triangle, mm is $=bb+gg$. Again the right-angl'd Triangles ARK, APM, are similar: Because taking the same Angle KAP from the equal Angles KAM, DAP, the remaining Angles KAR, PAM, shall be equal; and consequently $AR\left(\frac{ab+gz}{m}\right):RK\left(\frac{bz-bg}{m}\right)::AP\,(x):PM\,(y)$, from whence we get $z=\frac{aby+bgx}{bx-gy}$.

But the Right-angl'd similar Triangles HDF, HKS, give $HS=\frac{hz+hh}{n}$, and $KS=\frac{bz+bh}{n}$; and the Right-angl'd similar Triangles HFD, EBD, give $DH\,(n):DF\,(b)::DE\,(a-b):DB=\frac{ab-bb}{n}$.

And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Case, and of the opposite Sections in the second, must be parallel to the Lines AP, BQ; and the Ratio thereof to its Parameter, shall be the same as of EF (a) to FD (b); because the Fraction $\frac{b}{a}$ multiplying the Square xx expresses that Ratio.

When the Point K by its Motion along the indefinite right Line KF, comes to the Point O, wherein that Line meets the Circumference; then it is manifest, 1. That the Sides AM, BM, whose Point of Concurrence M does describe the Hyperbola BAM, do become parallel, 2. That they cut one another towards the opposite Side, during the Motion of the Point K along OL, that Part of the Line KF, falling within the Circle; and then they will again become parallel, when the Point K falls in L, after which they will begin again to cut one another towards the same Side. From whence it appears, that the Point M does describe the Hyperbola BAM during the Motion of the Point K along the two indefinite Parts of the right Line KF, that fall without the Circle; and the opposite Hyperbola, during the Motion of the Point K through OL, that Part of KF falling within the Circle.

FIG. 208. *Case* 3. Because here the indefinite right Line FK touches the Circumference of the Circle $BDAE$ in some Point F, it is manifest, that the Point D (in the two other Cases) does here coincide with the Point F; and so the Triangles DFG, DFH, do vanish; therefore we may use the Triangles DAB, BBE, for them, after the following manner.

Let the given Quantities AE be $= a$, $EB = b$, $EF = m$, $AF = g$, $BF = h$, $BI = c$, $AI = d$; and the unknown Quantities $FK = z$, $AP = x$, $PM = y$. Now the right-angled Triangles FKR, EFA, are similar; because the Angle KFR or TFA (vertically opposite thereto) made by the Tangent FT, and the Chord FA, is measur'd by half the Arc AF, as well as the Angle FEA; and therefore FE (m) : EA (a) :: KF (z) : $FR = \frac{az}{m}$. And EF (m) : FA (g) :: FK (z) : $KR = \frac{gz}{m}$. But the right-angled similar Triangles ARK, APM, do give AR or $AF + FR$ $\left(\frac{az+gm}{m}\right)$: RK $\left(\frac{gz}{m}\right)$ :: AP (x) : PM (y); from hence we get $z = \frac{gmy}{gx-ay}$: after the same manner, because EFB, FKS, are right-angled similar Triangles; therefore FS is $= \frac{bz}{m}$, and $KS = \frac{hz}{m}$; and because the right-angled Triangles BSK,

BQM, are similar; therefore BS or $BF - FS \left(\frac{bm - bz}{m}\right) : SK \left(\frac{bz}{m}\right)$ $:: BQ\,(x - c) : QM\,(y + d)$; from whence arises $x = \frac{bmy + bmd}{bx - cb + bd + by}$.

Now comparing the two Values of x, multiplying cross-wise, and putting the Terms in Order, and we shall have this Equation, $yy + dy - \frac{cgb}{ab + bg}y - \frac{dgb}{ab + bg}x = 0$, the Locus of which shall always be a Parabola, whose Axis is parallel to the right Lines AP, BQ; and the same may be constructed by Article the 310th.

Hence it is evident, 1. That the Locus of all the sought Points M shall be always a Conick Section, whose Axis, or one of the Axes, shall be parallel to the Lines AP, BQ; and particularly the same shall be an Ellipsis in the first Case, two opposite Hyperbola's in the second, and a Parabola in the third; and in the first and second Cases, the Axis which is parallel to AP shall have the same Ratio to its Parameter as EF to FD. 2. That in the first and third Cases, the two fix'd Points A and B, about which the moveable Angles KAM, KBM, revolve, do always fall on the same Side the Line FK: But in the second Case, those Points may not only fall on the same Side of that Line, but on both Sides thereof; because the Circumference of the Circle $ADBE$ upon which they are situate, is then cut into two Portions by the Line FK.

SCHOLIUM I.

372. 1. ANY right Line, as AM, passing through one of the fixed Points A or B, being given, the Point M wherein the same meets the Locus sought, may always be found after the following manner. Draw the right Line AK forming the Angle MAK with AM, equal to the given Angle revolving about the fixed Point A, and from the Point K wherein AK meets FK, through the fixed Point B, draw the Angle KBM equal to the other given Angle revolving about the other fixed Point B; then the Point M, wherein the Side BM of that Angle meets the Line AM, shall be the Locus sought. 2. When the Point K, in moving along the Line FK, is so situate, that the Side (AM) of the Angle KAM does coincide with the Line AB: Then it is plain that M the Point of Concurrence of the two Sides AM, BM, does fall on the Point B; and so the Locus of the Points M does pass through the fixed Point B; after the same Manner we prove that the same passes through the Point A. FIG. 206, 207, 208.

Hence if it be requir'd to describe the Conick Section being the Locus of all the sought Points M, without the Help of the foregoing FIG. 206.

Equa-

Equations, you need but draw (as in the Example) the right Lines AP, AI, and finding the Points wherein they meet the Section, and compleating the Rectangle, having those two Lines for the Sides thereof,
* Art. 176, and 178. then describe * an Ellipsis or two opposite Hyperbola's about the Rectangle, (according as FK falls without or within the Circle) whose Axis being parallel to AP may be to its Conjugate, as the Square of EF to the Square of DF. And if the Section be a Parabola, (which happens when the Line KF touches the Circle BDA;) then find a
Fig. 208. Point on the Line AI wherein the same meets the Section, and describe a Parabola (by Article 170) passing through that Point and the two given Points A, B; so that the Diameters thereof be parallel to the Lines AP, BQ.

SCHOLIUM. II.

Fig. 209. 373 WHEN the Point K in its motion along the Line FK, is so situate, that AM the Side of the Angle KAM, does coincide with AB; then it is manifest that the Point M falls on B;
* Art. 188. and also that BM the Side of the Angle KBM does touch * the Locus of the Point M in B, because in this Case M may be taken as being infinitely near to B. Whence if it be requir'd to draw a Line to touch that Locus in B, you need only draw a right Line AC through the Point A making an Angle (BAC) with BA equal to the given Angle KAM; and then a Line BD, making with BC the Angle CBD, equal to the other given Angle KBM. For the Side (BM) of that Angle, which does become BD, shall touch the Section in B. Understood the same with regard to the other fixed Point A.

Fig. 209. From hence arises a very easy way of describing the Conick Section being the Locus of all the Points M, without using the foresaid Equations, or even any manner of Calculus.

Through the fixed Point B draw the Tangent BD, and through the other fixed Point A draw AE parallel to that Tangent, and on
* Art. 372. AE find * the Point E wherein the same meets the Section, and bi-
* Art. 372. secting it in H, draw BH, upon which find * also the Point G wherein
* Art. 161, 162. the same meets the Section: Now if a Conick Section be describ'd * with the Diameter BG, and Ordinate HA or HE, the same shall be the Locus sought. For it is plain that the Line BG, which does bisect the Line AE terminated by the Section, and being parallel to the Tangent in B, shall be a Diameter thereof, and the Line AH, an Ordinate to that Diameter. Where it must be observed, that when the Point H falls between the Points B, G, the Section is an Ellipsis: when the same falls on either Side of those two Points, the Section is two opposite Hyperbola's; and finally, when the Line BG is infinite, the Section is a Parabola.

COROL-

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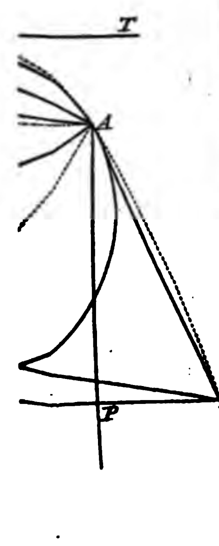
. 210.



Art. I
and 17

FIG.

FIG.

* *Art.*

FIG.

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COROLLARY I.

Means of the last Example we may describe a Conick FIG. 210.
ion of a given Species, through four given Points A, B
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t the Section be an Ellipsis, whose great Axis to its Para-
as the given Quantity a to b. Form the Triangle ABH,
hree of the given Points by straight Lines; and at the
nd B, let the Angles MAK, MBK, be made equal to
AH, RBA, the Complements of the Angles HAB,
wo right Angles, so that their Sides BM, AM, do in-
another in the fourth given Point M. This being done,
a Chord describe the Segment BDA of a Circle (on the
he Point M) containing an Angle equal to four right An-
the two Angles KAM, KBM; and about C the Centre
le, describe another Circle whose Radius CF may be to
CD of the former Circle, as $a+b$ to $a-b$; then from
nt of Concurrence of AK, BK, the Sides of the Angles
BK, draw the Tangent KF to this latter Circle. Now
t K be moved along the indefinite right Line FK, the Point
ence (M) of the two other Sides AM, BM, produced on
ide of the Points A, B, shall describe the Ellipsis required.
vident from what has been said in the first Case of the Ex-
at the Locus of the Points (M) shall be an Ellipsis, whose
s shall be to the Parameter as $EF(a)$ to $DF(b)$; and
that the same shall pass through the Points A,M,B,H, because
Point K is in G, the Side AM will fall in AH, and the
in BR.

n it is an Hyperbola or two opposite ones that is required to
ed through four given Points A, B, H, M, the great Axis
he Parameter in the given Ratio of a to b; then the Con-
vill be the same as above, only CF the Radius of the Circle
ck to the Circle $BDAE$, must here be to the Radius CD,
$a+b$.

Parabola be to be described through four given Points A,B,
escribe the Circle $BDAE$ as in the first Case, and from K
of Concurrence, draw a Tangent to that Circle, and the same
he indefinite right Line, along which, if the Point K be
e other Point of Concurrence M will describe the Parabola

there can be drawn two Tangents to a Circle from one
erefore we may describe two different Conick Sections, both
the Problem when the same is possible; for when the

 Point

Point *K* falls within the Circle, having *C F* for its Radius, it is plain then that the Problem is impossible.

The Conick Section sought may be described by means of its Axes according to Article 372, or else by help of one of its Diameters, and an Ordinate to the same, by Article 373.

Corollary II.

Fig. 211. 375. FROM the foregoing Example we have also a new way of describing a Conick Section passing through five given Points *A, B, H, M, N.* For joining any three of those Points, as *A, B, H*, by right Lines, and through the other two Points, *M, N*, and the fixed Points *A, B*, let the Angles *M A K, N A S*, pass, each equal to the Angle *H A G*, the Complement of the Angle *H A B* to two right Angles, and the Angles *M B K, N B S*, each equal to the Angle *A B R*, the Complement of the Angle *A B H* to two right Angles; then if through the Points of Concurrence *K, S*, the indefinite Line *SK* be drawn, and the Point *K* be moved along the same, it is manifest that the Point of Concurrence *M* by that Motion, shall describe the Conick Section requir'd; because it passes through the five given Points *A, B, H, M, N.*

The End of the Eighth Book.

BOOK

BOOK IX.

Of the Construction of EQUATIONS.

PROPOSITION I.

Problem.

376. TO *construct any given Equation, wherein the unknown Quantity is but of one Dimension.*

First, Let the unknown Quantity x be equal to one or more simple Fractions, as $\frac{ab}{c}$, or $\frac{abe}{cf}$, or $\frac{abeh}{cfg}$, *&c.* If we make $c : b :: a : l$, then it is plain, that this four Proportional l shall be $= \frac{ab}{c}$; and if we make $f : l :: e : m$, then m will be $= \frac{el}{f} = \frac{abe}{cf}$; and finally, making $g : m :: h : n$, and we shall have $n = \frac{mh}{g} = \frac{abeh}{cfg}$, by substituting $\frac{abe}{cf}$ for its Value m. Therefore the unknown Quantity x will be equal to l, or m, or n, *&c.* according as it was equal to $\frac{ab}{c}$, or $\frac{abe}{cf}$, or $\frac{abeh}{cfg}$, *&c.* Now by augmenting the Number of Proportions, (according to Necessity) and we shall always find a right Line equal to a given simple Fraction, be the Number of Dimensions of its Numerator what it will. From whence it appears, that a Line x may be always found equal to a Quantity compounded of several simple Fractions; for right Lines being found equal to each of those Fractions, we need only add them together, or substract them from each other, according to the Signs $+$ and $-$. For Example: Suppose it be requir'd to find a Line $x = a + \frac{ab}{c} + \frac{aab}{cf} - \frac{aacc}{b^3}$. Add the two Lines $h = \frac{ab}{c}$, and $l = \frac{aab}{cf}$ to the Line a, that so all three of them may make but one Line only, and from the Sum take the Line $m = \frac{aacc}{b^3}$, and what remains shall

be the Value sought of the unknown Quantity x, that is, x shall be $= a + b + l - m$.

unknown Quantity x be equal to one or more compound at is, Fractions whose Denominators consist of several Terms. t, according to the Directions above, find a Line equal to the Denominator divided by a Line taken at pleasure, when the Terms of the Denominator do not arise to above two Dimensions; by a Plane, when they arise not to above three; and by a Solid, when they do not arise to above four, &c. and by this means all the Terms of the Denominator may be brought into one only, which being substituted instead of them, shall change the compound Fraction into one or more simple Fractions, according as the Numerator consists of one or more Terms; and then if a Line be found (as above) equal to them, the same shall be that sought. This will be manifest by the following Examples.

It is requir'd to find a Line $x = \frac{age - bce}{bb + af}$. First, find a Line $m = f + \frac{bb}{a}$, that is, equal to the Denominator $af + bb$ divided by a; from whence $bb + af = am$, and then finding a Line $n = \frac{age - bce}{am} = \frac{ge}{m} - \frac{bce}{am}$; and we shall have the sought Line $x = n$. After the same manner, if it be requir'd to find a Line $x = \frac{a^3b + aacc - abcf}{aaf + ccf + bff}$, we must first find a Line $m = a + \frac{cc}{a} + \frac{bf}{a}$, that is, equal to the Denominator $aaf + ccf + bff$ divided by the Plane af; from whence we get $afm = aaf + ccf + bff$, and afterwards another Line $= \frac{a^3b + aacc - abcf}{afm} = \frac{aab}{fm} + \frac{acc}{fm} - \frac{bc}{m} = x$. The same must be understood of any other Example, which any one may form at pleasure.

Corollary I.

377. By means of this Proposition it is easy, 1. To find a simple Fraction $\frac{x}{a}$ or $\frac{a}{x}$, whose Denominator or Numerator a is given, equal to one or more simple or compound Fractions; for we need only find a Line x equal to the Line a multiply'd or divided by se Fractions. For Example; If it be requir'd to find a Fraction $= \frac{cc+ff}{af+cf} + \frac{aa}{gg}$, then it is plain, that we need only find a Line

=

$=\frac{acc+aff}{af+cf}+\frac{a^3}{gg}$. 2. To find a Plane ax, one of whose Sides a is given, equal to one or more Planes be they never so much compounded; for here we need only find a Line x equal to all those Planes divided by a. 3. To find a Solid aax or abx, whose Sides a, a, or a, b, are given, equal to several Solids; because here you need but find a Line x equal to all the Solids divided by the Square aa, or Plane ab. 4. To find a surd Solid a^3x, or $abcx$, whose three Sides a, a, a, or a, b, c, are given, equal to several surd Solids; because you need only find a Line x equal to all the surd Solids divided by the Cube a^3, or the Solid abc. And the same must be understood of several Products of five Dimensions, of six, &c. which may be always reduced to one only, whose Sides are all given except one.

COROLLARY. II.

378. HENCE, if it be requir'd to find a Square equal to several given Planes, the Planes must be brought into one, and then a mean Proportional found between the Sides of it; and then that mean Proportional shall be the Side of the Square requir'd. For Example: If it be requir'd to find a Square $xx=ss-\frac{ccee-eebb}{bb+af}$ (the Lines a, b, c, e, f, b, s, being given) first find a Line $m=\frac{ss}{e}-\frac{cce-ebb}{bb+af}$, that so we may have a Plane $em=ss-\frac{ccee-eebb}{bb+af}$, and then finding a mean Proportional x between e and m, the Sides of the Plane em; and it is plain, that xx is $=em=ss-\frac{ccee-eebb}{bb+af}$.

If it be requir'd to find a Line x, whereof the Square x^4 shall be equal to several given surd Solids; find (as above) a Square zz equal to all the given surd Solids divided by the Square aa either given or taken at pleasure. When this is done, find a mean Proportional as x, between the two Lines a and z, and that mean Proportional shall be the Line sought; for $xx=az$, and (squaring both Sides) $x^4=aazz$, that is, x^4 is equal to all the given surd Solids.

SCHOLIUM.

379. ALtho' the Method above laid down is general for all possible Cases, yet it is not always the most simple; therefore I shall here lay down some particular Examples, and resolve them more easy after a manner something different from the general Rule, which may serve as Rules for all the like Cases.

1. Let

1. Let x be $= \frac{abcc - aabb}{abc + c^3}$. First find a Line $m = \frac{ab}{c}$, and substituting cm for its Value ab, we shall have $x = \frac{c^3 m - ccmm}{ccm + c^3} = \frac{cm - mm}{m + c}$; from whence making $c + m : c - m :: m : n$, I find that this fourth Proportional n is $= x$. Therefore it is manifest, that we here find the Value of x by only two Proportions, whereas by the general Method there must have been three at least.

2. Let x be $= \sqrt{aa+bb}$. Make a right-angl'd Triangle, one of whose Sides let be $= a$, and the other $= b$; then the Hypothenuse thereof shall be the Value of x. If it be requir'd to find a Line $x = \sqrt{aa-bb}$, then find a mean Proportional as x, between the two Lines $a + b$ and $a - b$; for the Square thereof must be equal to the Product $aa - bb$ of the Extremes. Or else make a right-angled Triangle, whose Hypothenuse let be $= a$, and one of the Sides $= b$; then the other Side shall be the Value of x.

3. Let xx be $= ss + 4ee - \frac{4ccee}{aa}$. Assume m equal to the Hypothenuse of a right-angled Triangle having one Side $= s$, and the other $= 2e$, and finding another Line $n = \frac{2ce}{a}$, we have $xx = mm - nn$, and $x = \sqrt{mm-nn}$, which may be resolved in the same manner as in the last Example where x was $= \sqrt{aa+bb}$.

Lastly, Let xx be $= ss - \frac{ccee - eebb}{bb + af}$. Assume a mean Proportional between a, f, the Sides of the Plane af, that so we may have $ll = af$, then find a Square $mm = bb + ll$, and another Square $nn = cc + bb$, by means of two right-angled Triangles, as in the second Example, and (by Substitution) we have $xx = ss - \frac{eenn}{mm}$; and finally, finding a Line $g = \frac{en}{m}$, and there arises $x = \sqrt{ss-gg}$, which may be resolved as above.

PROPOSITION II.

Problem.

380. *TO find the Roots of all kinds of Equations of the second Degree,* [*or of two Dimensions.*]

All Equations of the Second Degree may be reduced to one of the following Forms $xx \mp ax - bb = 0$, or $xx + ax + bb = 0$, by finding a * Line a equal to all the known Quantities that do multi-

*Art. 376.

multiply the unknown one x, and a * Square bb equal to all the known *Art. 378.
Rectangles. This being premised.

1. Let $xx + ax - bb$ be $= o$. Make a right-angl'd Triangle CAB, Fig. 212.
one of whose Sides CA let be $= \frac{1}{2}a$, and the other Side $AB = b$, and drawing the Hypothenuse BC, and producing the same beyond C; about C as a Centre with the Radius CA, describe a Circle cutting BC in the Points E, D. I say the right Lines BD, BE, are the two Roots of the proposed Equation $xx \mp ax - bb$; BE being the affirmative Root, and BD the negative Root of the Equation $xx + ax - bb = o$; and contrariwise BD the affirmative, and BE the negative Root of the Equation $xx - ax - bb = o$.

For making $BE = x$, and we shall have BD or $BE + ED = a + x$; and if BD be made $= -x$, then BE or $BD - ED$ shall be $= -x - a$. And so in both Cases $DB \times BE$ is $= xx + ax = \overline{AB}^2 (bb)$ by the Nature of the Circle, that is, $xx + ax - bb$ is $= o$. And contrariwise, if BD be made $= x$ or $BE = -x$, we shall find $DB \times BE = xx - ax = bb$ or $xx - ax - bb = o$.

2. Let $xx \mp ax + bb$ be $= o$. Make a right-angl'd Triangle (as Fig. 3.
in Case 1.) CAB, having one Side $CA = \frac{1}{2}a$, and the other $AB = b$; and the indefinite right Line BD being drawn parallel to AC, about the Centre C with the Radius CA describe a Circle cutting the Line BD in the Points E, D. I say the right Lines BE, BD, are the Roots of the proposed Equation $xx \mp ax + bb = o$; *viz.* the two affirmative Roots of $xx - ax + bb = o$, and the two negative ones of $xx + ax + bb = o$.

For compleating the Semi-circumference $AEDH$, and drawing EF, DG, parallel to AB; then if BE or AF be made $= x$, we shall have $AF \times FH = ax - xx = \overline{FE}^2 (bb)$ by the Nature of the Circle. In like manner, if BD or AG be made $= x$, we shall have $AG \times GH = ax - xx = \overline{GD}^2 (bb)$: That is, $xx - ax + bb = o$ in both Cases. If BE or AF be $= -x$, and BD or $AG = -x$, then shall $AF \times FH$ and $AG \times GH$ be $= -xx - ax = \overline{FE}^2$ or $\overline{GD}^2$ (bb); that is, $xx + ax + bb = o$.

If the Circle, whose Centre is C, and Radius CA, does not cut or touch the parallel BD (which happens always when AB exceeds CA); then both the Roots of the Equation will be imaginary; but if the Circle touches the same in one Point, the two Roots BE, BD, do each become equal to the Radius CA.

SCHOLIUM.

381. WHEN the unknown Quantity in an Equation hath only four and two Dimensions; then that Equation may be always brought to another wherein the unknown Quantity arises no higher

higher than the second Degree; and so Equations of this kind may be taken for those of the second Degree.

FIG. 214. For Example, let $z^4 - aazz - aabb$ be $= o$. Suppose an unknown Quantity x to be such that the Rectangle under the same and the known Quantity a be equal to the Square zz; that is, let ax be $= zz$. Then substituting ax for zz, and $aaxx$ for z^4, and the given Equation $z^4 - aazz - aabb = o$, will be brought to this $xx - ax - bb = o$, wherein the unknown Quantity x arises no higher than the second Degree. And if the Roots (x) thereof be found as above, and mean Proportionals be found between the known Quantity a, and the Values of those Roots; then it is evident that those mean Proportionals shall express the sought Values of the unknown Quantity z: Because zz is $= ax$.

PROPOSITION III.

Problem.

382. *To find the Roots of Equations of the second Degree another way without necessarily charging the last Term into a Square.*

FIG. 215. 1. Let $xx \mp ax - bc$ be $= o$, wherein b exceeds c. Describe any Circle ABD having its Diameter not less than the given Quantities a and $b-c$, and within this Circle, inscribe two Chords $AB = a$, $AD = b-c$, both from any Point A thereof: And producing AD to F, so that $DF = c$, about the Centre C with the Radius CF, describe another Concentrick Circle cutting the Chords AD, AB, (produced) in the Points F, E, G, H. I say AG is the affirmative, and AH the negative Root of $xx + ax - bc = o$; and contrariwise AG the negative, and AH the affirmative Root of the Equation $xx - ax - bc = o$.

For AF or $AD + DF = b$, and DF or $AE = c$, and making AG or $BH = x$, we shall have $AH = a + x$. And by the Property of the Circle $EGFH$, the Rectangle $EA \times AF (bc) = GA \times AH (xx + ax)$. Now if AH be made $= -x$, we shall have AG or BH or $AH - AB = -x - a$, and consequently $GA \times AH = xx + ax$ as before. Therefore whether AG be $= x$, or $AH = -x$, we shall always have $xx + ax - bc = o$. After the same manner we prove that AG is the negative, and AH the affirmative Root of $xx - ax - bc = o$.

FIG. 216. 2. Let $xx \mp ax + bc$ be $= o$. Describe any Circle ABD, whose Diameter is not less than the given Quantities a and $b + c$, and within the same inscribe two Chords $AB = a$, $AD = b + c$, both from any Point A thereof: Then in AD assume $DF = c$, and about the Centre C with the Radius CF describe another Concentrick Circle, cutting the Chords AD, AB, in the Points F, E, G, [illegible]. I say

say AG and AH, are the two affirmative Roots of $xx - ax + bc = 0$, and the two negative Roots of $xx + ax + bc = 0$. This is demonstrated after the same manner as in the first Case.

If the Circle, whose Radius is CF, does neither cut nor touch the Line AB; then the two Roots of the Equation shall be imaginary.

ADVERTISEMENT.

All the Contrivance that I make use of, in the Construction or Investigation of the Roots of an Equation, consisting of but one unknown Quantity, lies in bringing a new unknown Quantity into that Equation; that thereby several Equations may be had, each containing the two unknown Quantities; and moreover may be such, that any two of them do contain together all the known Quantities of the propos'd Equation; because otherwise, when the new unknown Quantity is struck out, the propos'd Equation will not again arise. Then among those Equations I pick out two of the most simple, and construct their Loci separately, and the Intersection of those Loci will give the Roots sought. Now that unknown Quantity must be so taken, that the Loci of the Equations arising from the propos'd Equation, be the most simple possible. For Example: If the Equation be one of the fourth Degree, the Loci of the two Equations must not exceed the second Degree: Among which Loci there must be always a Circle, as being most simple, and also a Parabola, Equilateral Hyperbola, &c. All this will fully appear in the following Lemmata and Propositions.

A FUNDAMENTAL LEMMA *for the Construction of Equations of the third and fourth Degree, by means of a Circle and a given Parabola.*

383. LET there be a propos'd Equation $x^4 + 2bx^3 + acxx - aadx - a^3f = 0$, wherein x is unknown, and a, b, c, d, f, are known; and suppose another unknown Quantity y to be such, that the Rectangle under the same, and the known one a, be equal to the Rectangle under $x + b$ and x; and from hence we have the following Equations.

1. $ay = xx + bx$, both Sides of which being squared, and there arises $x^4 + 2bx^3 + bbxx = aayy$, and substituting $aayy - bbxx$ in the Equation propos'd for its Value $x^4 + 2bx^3$, and the same shall be chang'd into this Equation.

2. $yy - \frac{bb}{aa}xx + \frac{c}{a}xx - dx - af = 0$, wherein substituting for xx its Value $ay - bx$ found by means of the first Equation, 1. in $-\frac{bb}{aa}xx$. 2. in $\frac{c}{a}xx$. 3. in $-\frac{bb}{aa}xx + \frac{c}{a}xx$, and the following three Equations shall be had.

$3.\ yy - \frac{bb}{a}y + \frac{b^3x}{aa} + \frac{c}{a}xx - dx - af = 0.$

$4.\ yy - \frac{bb}{aa}xx + cy - \frac{bc}{a}x - dx - af = 0.$

$5.\ yy + cy - \frac{bb}{a}y - \frac{bc}{a}x + \frac{b^3}{aa}x - dx - af = 0.$ If the first Equation $xx + bx - ay = 0$, be taken from this fifth Equation, and afterwards added to it, then we shall have these two others, viz.

$6.\ yy + cy - \frac{bb}{a}y + ay - xx - bx - \frac{bc}{a}x + \frac{b^3}{aa}x - dx - af = 0.$

$7.\ yy + cy - \frac{bb}{a}y - ay + xx + bx - \frac{bc}{a}x + \frac{b^3}{aa}x - dx - af = 0.$

Now if for the unknown Quantities x and y, there be taken two right Lines AP, PM, making any Angle APM with each other;
* Art. 310. then it is evident, * that the Locus of the first Equation is a Parabola; that of the second may be a Parabola, Ellipsis, or Hyperbola, according as bb is equal, less, or greater than ac; that of the third, an El-
* Art. 328, lipsis, which does become * a Circle when c is $= a$, and the Angle
and 329. APM is a right Angle; that of the fourth, an Hyperbola, which
* Art. 335, does become equilateral when * b is $= a$; that of the fifth is also a
and 336. Parabola; that of the sixth an Equilateral Hyperbola; and lastly, the Locus of the seventh is a Circle, when the Angle APM is a right Angle.

Scholium I.

384. If $2bx^3$ had been negative in the propos'd Equation, then the Signs of all the Terms wherein b is found of odd Dimensions in all the Equations must have been changed; and if the second Term was wanting, then all the Terms affected with b must have been struck out. The same must be understood with regard to the other Terms of the proposed Equation in respect of the Letters c, d, f, contain'd in them. But it must be observed here, that in all the different Alterations that can happen, the Locus of the first Equation shall be a Parabola, that of the sixth an Equilateral Hyperbola; and lastly, that of the seventh a Circle, when the Angle APM is a right Angle.

Scholium II.

385. The Reason why we have chosen the first Equation $xx + bx = ay$, rather than $xx - bx = ay$, or simply $xx = ay$, is, because when both Sides thereof are squared, the two first Terms of one Side are the same as the two first Terms of the proposed Equation

x^4

$x^4 + 2bx^3$, &c. and so they may be made to destroy one another. And by that means we shall get a new Equation, whose Locus is no higher than the second Degree, which being combin'd different ways with the first, gives other Equations (as appears above) whose Loci, not being higher than the second Degree, may be easily constructed, because the Plane xy is not contain'd in those Equations; among which the Locus of the last is always a Circle, supposing the unknown Quantities x and y to make a right Angle with one another.

PROPOSITION IV.

Problem.

386. *To find the Roots of the proposed Equation* $x^4 + 2bx^3 + acxx - aadx - a^3f = 0$, *by means of a Parabola and Circle.* FIG. 217.

Assume two right Lines AP, PM, making a right Angle APM with one another, for the unknown and indeterminate Quantities x and y; and then construct * the Parabola which is the Locus of the first Equation of the Lemma, and afterwards the Circle which is the Locus of the seventh; and by means of the Intersection of these two Loci, the different Values of the unknown Quantity x which shall be the Roots of the proposed Equation, may be found. This may be done after the following Manner. *Art. 310.

In the Line AP produced on the other Side of A, assume $AD = \frac{1}{2}b$, and through the Point D draw a Parallel to PM, in which Parallel take $DC = \frac{bb}{4a}$ on the contrary Side of AP with regard to PM, and with CD as an Axis, (the Point C its Origin,) and a Line equal to a for its Parameter, describe a Parabola MCM. This being done, through the fixed Point A draw AQ parallel to PM, and in the same assume $AB = \frac{1}{2}a + \frac{bb}{2a} - \frac{1}{2}c = \mp g$ for brevities sake, and parallel to AP draw the right Line $BE = \frac{1}{2}d \pm \frac{bg}{a}$, *viz.* $-\frac{bg}{a}$ when AB is $= +g$, that is, when the Value of AB is affirmative, and $+\frac{bg}{a}$ when $AB = -g$; observing to take or draw both the Lines AB, BE, on the same Side AP as PM is, when their Values are positive, and on the contrary Side when the same are negative. Lastly, calling EA, m; about the Centre E, with the Radius $EM = \sqrt{mm \mp af}$ describe a Circle: Then if Perpendiculars MP be drawn from the Points M wherein the Circle cuts the Parabola, to the Line AP, the parts (AP) of that Line shall denote the Roots of the Equation, the affirmative Roots being on the same Side A as PM was supposed to be in the Constructing the Parabola, and the negative ones, on the contrary Side.

 For

For producing MQ (parallel to AP) until it meets the Axis CG in the Point L, we have ML or $AP + AD = x + \frac{1}{2}b$, CL or $MP + DC = y + \frac{bb}{4a}$; and by the Property of the Parabola $\overline{ML}^2 = CL \times a$, that is, $xx + bx + \frac{1}{4}bb = \frac{1}{4}bb + ay$, or $xx + bx = ay$, which is the first Equation of the Lemma. Now if EB be produced until it meets PM in R, and the Radius EM be drawn, then because ERM is a right-angled Triangle, the Square $\overline{EM}^2$ shall be $= \overline{ER}^2 + \overline{RM}^2 = \overline{EB}^2 + 2EB \times BR + \overline{BR}^2 + \overline{PM}^2 - 2AB \times PM + \overline{AB}^2 = \overline{EB}^2 + \overline{BA}^2 + af$ by Construction, and striking out the Squares $\overline{EB}^2$, $\overline{BA}^2$ from both Sides, substituting $a + \frac{bb}{a} - c$, for $2AB$, $\frac{2bg}{a} - d$, or $b + \frac{b^3}{aa} - \frac{bc}{a} - d$, for $2BE$, and x, y, for BR, or AP and PM, and then we shall get the seventh Equation $yy + cy - ay - \frac{bb}{a}y + xx + bx + \frac{b^3}{aa}x - \frac{bc}{a}x - dx = af$, wherein if $\frac{xx + bx}{a}$ be substituted for its Value y, and the Square thereof for yy, then we shall get again the proposed Equation $x^4 + 2bx^3 + acxx - aadx - a^3f = 0$. From whence it appears, that the Line AP expresses an affirmative Root of this Equation.

If $-x$ and $-y$ be taken for AP and PM, when these Lines do fall the contrary way to that they are suppos'd to fall in the Construction; then the first Equation will be always found by the Property of the Parabola, and the seventh by the Property of the Circle. Therefore, &c.

COROLLARY I.

387. HENCE the foregoing Construction may be made general for all Equations of the third and fourth Degree; and a Parabola having the Parameter of its Axis equal to the given Line a, shall be always used in that Construction. If you, (1.) multiply an Equation
*Art. 376. of the third Degree by the Root x, and take a Line * $2b$ equal to all
*Art. 377. the Lines that do multiply x^3, a Plane * ac equal to all those that multiply xx, a Solid aad equal to the Solids multiplying x; and lastly, a third Solid a^3f equal to the known Terms of the Equation. (2.) And change the Signs wherein b is found of odd Dimensions, in the Values of the Lines AD, DC, AB, BE, EM, which determine the Construction of the Parabola and Circle, if $2bx^3$ be negative in the proposed Equation, because the same is positive in the Problem; and more-

moreover, destroy all the Terms affected with b, if the Term $2bx^3$ be wanting, because then b is $=0$; and do the same with regard to the Terms wherein c, d, f, do happen. (3.) And take or draw the Lines when affirmative towards, or on the same Side as PM; and when negative, the contrary way. Then we shall have $AD = \pm \frac{1}{2}b$, *viz.* $-\frac{1}{2}b$ when $2bx^3$ is affirmative, and $+\frac{1}{2}b$ when the same is negative; $AB = \frac{1}{2}a + \frac{bb}{2a} \pm \frac{1}{2}c = \pm g$, *viz.* $-\frac{1}{2}c$ when $acxx$ is affirmative, and $+\frac{1}{2}c$ when the same is negative; $BE = \pm \frac{bg}{a} \mp \frac{1}{2}d$, *viz.* $-\frac{bg}{a}$ when AB is $=+g$, and when $2b^3$ is affirmative, or else when AB is $=-g$, and $2bx^3$ is negative; and contrariwise $+\frac{bg}{a}$ when $AB = +g$, and $2bx^3$ is negative, or else when $AB = -g$, and $2bx^3$ is affirmative (that is, $-\frac{bg}{a}$ when the Values of AB, and AD, are one positive, and the other negative, and $+\frac{bg}{a}$ when their Values are either both positive, or both negative) as likewise $+\frac{1}{2}d$ when $aadx$ is negative, and $-\frac{1}{2}d$ when the same is positive; and lastly, $EM = \sqrt{mm \mp af}$, *viz.* $+af$ when a^3f is negative, and $-af$ when the same is positive. From hence arises the following Construction, which is general for all Cases.

A Parabola MCM whose Axis is the Line CG having a Line equal to a for its Parameter, being given, and the proposed Equation being reduced to this Form $x^4 \mp 2bx^3 \mp acxx \mp aadx \mp a^3f = 0$, draw a Line AB parallel to the Axis CG distant therefrom by the Quantity $\frac{1}{2}b$, on the right Side of the Axis when $2bx^3$ is affirmative in the proposed Equation, and on the left Side of the same when $2bx^3$ is negative. Through the Point A wherein the Line AB meets the Parabola, draw AD perpendicular to the Axis CG, and in the Axis assume $DF = \frac{1}{2}a$, $FG = 2CD$ (always from D the contrary way to the Origin C) and GK towards C, when $acxx$ is positive; but the contrary way, when the same is negative. Then through the determinate Points A, F, draw an indefinite right Line AF, and thro' the Point K a perpendicular to the Axis meeting AF in H; and in this Perpendicular take $HE = \frac{1}{2}d$ on the right Side thereof when $aadx$ is negative, and on the left Side when the same is affirmative. This being done about the Centre E, and with the Radius $EM = AE$, when the Term a^3f is wanting in the given Equation; that is, when the same is but of the third Degree; but when it is of the fourth, call AE, m, and take the Radius $EM = \sqrt{mm \mp af}$, *viz.* $+af$ when a^3f is nega- FIG. 217.

tive,

tive, and $-af$ when the same is affirmative. Lastly, Drawing Perpendiculars MQ from the Points M, wherein the Circle meets the given Parabola, to the Line AB; and these Perpendiculars shall be the Roots of the given Equation; those that fall on the right Side of the Line AB, being the affirmative Roots; and those falling on the left, the negative Roots.

For producing HK till it meets the Line AB in the Point B, then by Construction AD is $= \mp \frac{1}{2}b$, *viz.* $-\frac{1}{2}b$, when $2bx^3$ is affirmative, and $+\frac{1}{2}b$ when the same is negative; but by the Property of the Parabola $CD = \frac{bb}{4a}$. Therefore DG or $DF + FG = \frac{1}{2}a + \frac{bb}{2a}$, and DK or $AB = \frac{1}{2}a + \frac{bb}{2a} \pm \frac{1}{2}c = \mp g$, *viz.* $-\frac{1}{2}c$ when $+acxx$ is affirmative, and $+\frac{1}{2}c$ when the same is negative; and it must be here observed, that the Point B falls on the same Side as PM, when $AB = +g$, that is, when the Value thereof is positive, and on the contrary Side when the Value is negative. Now because the Triangles ADF, ABH, are similar, therefore $DF(\frac{1}{2}a) : DA(\pm\frac{1}{2}b) :: AB(\mp g) : BH = \pm\frac{bg}{a}$, *viz.* $+\frac{bg}{a}$, when the Values of AD and AB are both positive or both negative, and $-\frac{bg}{a}$ when the Value of one is positive, and of the other negative. And therefore $BE = \pm\frac{bg}{a} \pm \frac{1}{2}d$, *viz.* $-\frac{1}{2}d$ when $aadx$ is negative, and $+\frac{1}{2}d$ when $aadx$ is affirmative; but you must observe that the Point E will fall on the same Side as PM, when the Value of BE is positive, and on the opposite Side when the same is negative. Hence it appears that the Centre E of the Circle shall be always determined, as requsite in all the possible Cases, by this Construction.

If the second Term $2bx^3$ be wanting in the proposed Equation, then it is manifest that the Lines AB, AF, will fall in the Axis CG, in such manner that the Points A, D, shall coincide with C the Origin thereof; since $b = o$. And consequently the Point G will coincide with the Point F, and the Points H and B with the Point K: And so FIG. 218. the general Construction in this Case will be much simpler than that above. For here you need but take $CF = \frac{1}{2}a$ in the Axis of the Parabola, always within the same, and $FK = \frac{1}{2}c$ from F towards the Origin C when $acxx$ is affirmative, and the contrary way when the same is negative; and then draw $KE = \frac{1}{2}d$ perpendicular to the Axis on the left Side thereof when $aadx$ is affirmative, and on the right when the same is negative; and proceed afterwards as in the general Construction, observing that EC is here $= m$.

In like manner if the Term $aexx$ be wanting, then the Point K shall fall on the Point G; and if the Term $aadx$ be wanting, the Centre E of the Circle shall fall in H. FIG. 217.

COROLLARY II.

388. THERE may be had yet a more simple Construction for Equations of the third Degree that have their second Terms, in multiplying them by the unknown Quantity plus or minus, the known Quantity of the second Term, *viz.* plus that Quantity when the second Term is affected with the Sign $-$; and minus that Quantity when the same is affected with the Sign $+$; for in doing thus, we get an Equation of the fourth Degree wanting the second Term. For Example, to find the Roots of this Equation of the third Degree, $x^3 - bxx + apx + aaq = 0$: Multiply it by $x + b$, and then you will have the following Equation of the fourth Degree $x^4 \begin{array}{l} + apxx + aaqx + aabq = 0, \\ - bbxx + abpx \end{array}$ wanting the second Term; now using the Construction already laid down for these Equations that want the second Term, and then we shall have $CK(\frac{1}{2}a \pm \frac{1}{2}c) = \frac{1}{2}a + \frac{bb}{2a} - \frac{1}{2}p$, KE. $(\frac{1}{2}d) = \frac{1}{2}q + \frac{bp}{2a}$, and the Radius of the Circle $EM = \sqrt{mm - bq}$: From whence arises the following Construction.

Draw a Parallel to the Axis CD distant therefrom to the left by a Line equal to b, and meeting the Parabola in the Point A, also draw the Line CA through C the Origin of the Axis, and upon O the middle of CA raise the indefinite Perpendicular OG meeting the Axis in the Point G. This being done in the Axis assume $GK = \frac{1}{2}p$ from G towards C, and through the Point K draw a Perpendicular to the Axis meeting the Line OG in the Point H, in which perpendicular produced towards H, take $HE = \frac{1}{2}q$, and about the Centre E, with the Radius EA describe a Circle. I say this Circle shall cut the Parabola in Points $(M,)$ from whence letting fall Perpendiculars MQ to the Axis; and those Perpendiculars on the right of the Axis shall be the affirmative, and on the left the negative Roots of the proposed Equation $x^3 - bxx + apx + aaq = 0$. FIG. 219.

For if the right Lines AD, OL, be drawn perpendicular to the Axis; then by Construction we have $AD = b$, and by the Property of the Parabola $CD = \frac{bb}{a}$. Therefore since CA is bisected in O, the similar Triangles CAD, COL, shall give $OL = \frac{1}{2}b$, $CL = \frac{bb}{2a}$; and because the right angl'd Triangles CLO, OLG, are similar,

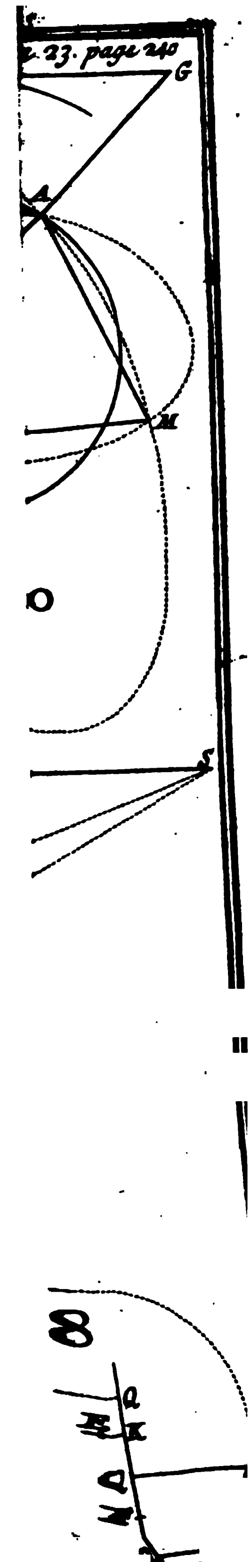
therefore $CL\left(\frac{bb}{2a}\right) : LO\,(\frac{1}{2}b) :: LO\,(\frac{1}{2}b) : LG = \frac{1}{2}a$, and consequently CK or $CL + LG - GK = \frac{1}{2}a + \frac{bb}{2a} - \frac{1}{2}p$. Moreover, because the Triangles GLO, GKH, are similar, therefore KH shall be $= \frac{bp}{2a}$, and $KH + HE$ or $KE = \frac{1}{2}q + \frac{bp}{2a}$, which tends to the left of the Axis, as is prescribed in the Construction, when $aadx$ is affirmative. Therefore the Point E is the Centre of the Circle, whose Intersections with the given Parabola shall determine all the Roots of the Equation of the fourth Degree $x^4 + apxx$, &c. And because the Roots of this Equation are the Roots of the proposed Equation $x^3 - bxx + apx + aaq = 0$, together with a negative Root $AD\,(b)$; whence the Circle must pass through the Point A. Therefore, &c.

We can prove likewise by Calculation, that EA is the Radius of the Circle sought. For if EB be drawn parallel to the Axis, then because the Triangles EAB, EKC, are right-angled, the Squares of the Hypothenuses $\overline{EA}^2 = \overline{EB}^2 + \overline{BA}^2$, and $\overline{EC}^2 = \overline{CK}^2 + \overline{KE}^2$, and consequently it must be proved, that $\overline{EB}^2 + \overline{BA}^2 = \overline{EK}^2 + \overline{KC}^2 - bq$, because we must take $EM = \sqrt{mm - bq}$. But substituting on both Sides, instead of those Lines, their Analytick Values, and these same Quantities will arise, as they must if the Radius sought EM be $= EA$.

Now to render the aforesaid Construction general, you must, (1.) Draw the Parallel to the Axis, being distant therefrom by a Line equal to b, on the left Side the same, when bxx is negative in the proposed Equation; and on the right Side, when the same is affirmative. (2.) And you must take $GK = \frac{1}{2}p$ in the Axis from G towards its Origin C, when apx is affirmative; and the contrary way, when the same is negative. (3.) And $HE = \frac{1}{2}q$ must be taken to the left, when aa is affirmative; and to the right, when the same is negative.

SCHOLIUM.

389. HENCE it is necessary to observe, 1. If the Circle does cut the given Parabola in two Points only, then the proposed Equation shall have but two real Roots, when it is one of the fourth Degree; and but one real Root, when it is an Equation of the third Degree, and the two others imaginary; as in *Fig.* 219. where the Circle cuts the Parabola in two Points A, M, only; the Equation $x^4 + apxx - bbxx$, &c. has but two real Roots AD, MQ, which are both negative ones, because they fall on the left Side the Axis. 2. If the Circle does neither cut nor touch the Parabola (which cannot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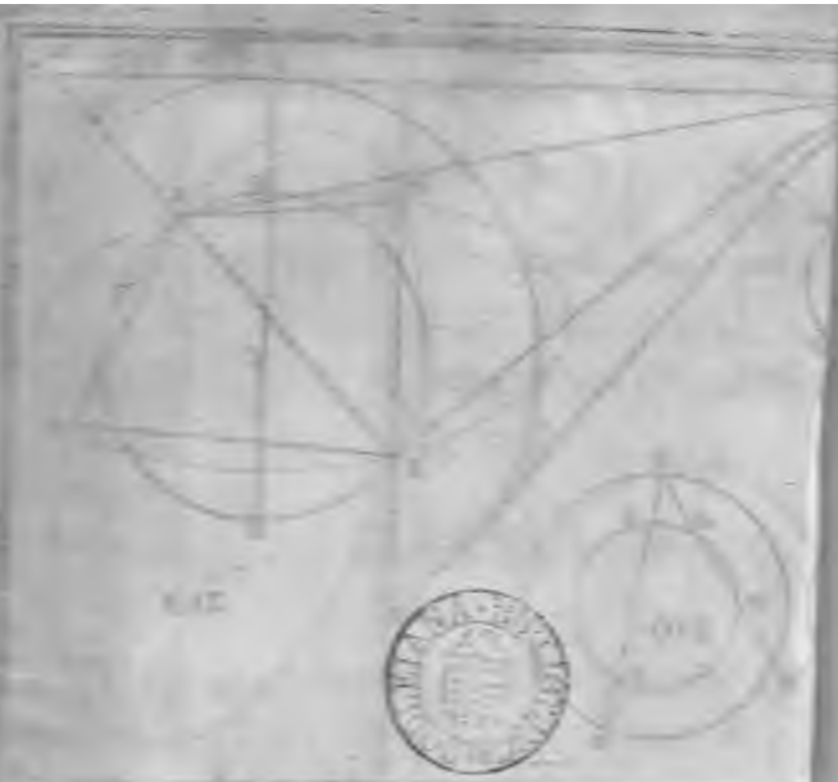


happen when the Equation is one of the third Degree, as appears by the foregoing Constructions) then the four Roots shall be imaginary. 3. If the Circle touches the Parabola, then the proposed Equation will have two Roots, each being equal to the Perpendicular drawn from the Point of Contact; as appears from hence, *viz.* that a Circle touching a Parabola, may be consider'd as cutting the same in two Points infinitely near to each other, which do coincide in the Point of Contact: But then the proposed Equation may be brought down to one of the second Degree by common Algebra, and so there will be no Necessity of constructing a Parabola for finding the Roots.

SCHOLIUM II.

390. IF Regard be had to what is demonstrated in Algebra, *viz.* That in any Equation wanting the second Term, and having all real Roots, the Sum of the affirmative Roots is equal to the Sum of the negative ones; we may get the following Theorem.

If a Circle cuts a Parabola in four Points M, and Perpendiculars, as MQ, be drawn from them to the Axis CF; I say, the Sum of the Perpendiculars that fall on the right Side of the Axis, shall be equal to the Sum of the Perpendiculars that fall on the left Side. FIG. 21

For if CF be taken in the Axis of the Parabola from C, its Origin within the Parabola, equal to the Parameter, which call a; and if EK be drawn from E the Centre of the Circle perpendicular to the Axis, and you make $FK = \frac{1}{2} c$, $KE = \frac{1}{2} d$, $\overline{EC}^2 - \overline{EM}^2 = af$; then it is plain * from the Construction at the End of *Cor.* 1. that the Perpendiculars MQ shall be the Roots of the Equation $x^4 - acxx + aadx + a^3 f = o$ wanting the second Term, *viz.* those falling on the right Side the Axis being the affirmative Roots, and those on the left, the negative ones; therefore, *&c.* *Art. 38

If the Circle passes through C the Origin of the Axis, then it is plain, that one of the Perpendiculars MQ will become $= o$, and so the Perpendicular on one Side the Axis shall be equal to the Sum of the Perpendiculars on the other.

If the Circle touches the Parabola in one Point, and cuts the same in two others, then you must assume twice the Perpendicular drawn from the Point of Contact, because that Circle may be look'd upon * as cutting the Parabola in two Points infinitely near to one another, which do coincide in the Point of Contact. *Art. 38

SCHOLIUM III.

391. BEcause in Geometry there cannot be suppos'd Products of more than three Dimensions, since a Solid, which is the most compounded, has but three; therefore you may divide all the Terms

 of

of a proposed Equation exceeding the third Degree by any given Line (taken at pleasure) rais'd to a Power, less by Unity than the Power of the highest Term thereof; and by this Means the Equation will not be at all embarrass'd, and each Term thereof shall express right Lines only. For Example: Let there be an Equation of the fourth Degree, *viz.* $x^4 + 2bx^3 + acxx - aadx - a^3f = 0$; divide the same by a^3, and there will arise $\frac{x^4}{a^3} + \frac{2bx^3}{a^3} + \frac{cxx}{aa} - \frac{dx}{a} - f = 0$, each of whose Terms hath but one Dimension, and consequently do express right Lines only. We commonly chuse that Line which is found repeated most times in all the Terms of the proposed Equation, as the Line a is here; and even sometimes it is understood in esteeming the same as Unity, which will produce no Alteration in the Quantities it multiplies or divides; so making $a = 1$, and then we may write $x^4 + 2bx^3 + cxx - dx - f = 0$, instead of $x^4 + 2bx^3 + ac$ $xx - aadx - a^3f = 0$, or of $\frac{x^4}{a^3} + \frac{2bx^3}{a^3} + \frac{cxx}{aa} - \frac{dx}{aa} - f = 0$. The same must be understood of Equations of the fifth Degree, sixth Degree, *&c.*

SCHOLIUM IV.

392. AFter the Circle, which is the Locus of the last Equation of the Lemma, is constructed, if a Conick Section, which is the Locus of any other of the Equations of that Lemma, be constructed; then the Intersections of those two Loci shall determine the Roots of the proposed Equation; because making the unknown Quantity y vanish by means of their Equations, the proposed Equation will be again gotten.

From whence it is evident, that that Equation may be constructed, 1. By means of a Circle and Equilateral Hyperbola, being the Loci of the seventh and sixth Equations of the Lemma. 2. By means of a Circle and an Ellipsis, whose Axis (being parallel to AP) is to the Parameter as a to c, using the seventh and third Equations. 3. By means of a Circle and Hyperbola, whose Axis (being parallel to AP) is to the Parameter as aa to bb, using the seventh and fourth Equations. And because the Line a, by means of which all the Quantities multiplying xx are reduced to the Expression ac, all the Quantities multiplying x to the Expression aad, and all the Quantities multiplying the known Terms to the Expression a^3f, is a Quantity at pleasure; therefore, if for a be taken for an infinite Number of different Lines, the proposed Equation may be constructed by means of an infinite Number of Circles, Ellipses, or Equilateral or not Equilateral Hyperbola's, all different between themselves.

It appears, (in Article 387.) that if the arbitrary Quantity a being the Parameter of the Axis of a given Parabola be taken for Unity, then using the first and seventh Equation, the propos'd Equation may be constructed by means of a Circle and a given Parabola: And here I shall shew how to take the Quantity a such, that the Equation may be constructed by means of a Circle, and an Ellipsis or an Hyperbola similar to a given Ellipsis or Hyperbola. For the Ratio of the Axes being given, by supposition, the Ratio of the Axis (parallel to AP) to its Parameter shall be given also. Therefore if that given Ratio be called $\frac{n}{m}$; we shall have $\frac{c}{a}=\frac{n}{m}$, (when the Section is an Ellipsis) and so $aa=\frac{acm}{n}$; from whence if for the arbitrary Quantity a (being Unity) there be taken the Root of a Square aa equal* to the known Quantity ac which multiplies xx in the given Equation, and is multiplied by $\frac{m}{n}$; the Equation may be constructed from the seventh and third Equations, by means of a Circle and an Ellipsis, whose Axis being parallel to AP, is to its Parameter as m to n, because $\frac{n}{m}=\frac{c}{a}$: But when the Section is to be an Hyperbola, we have $\frac{n}{m}=\frac{bb}{aa}$, and therefore $a=b\sqrt{\frac{m}{n}}$; whence if that Value be taken for unity (a) and the Equation be constructed from the seventh and fourth Equations, then the Axis of the Hyperbola (being parallel to AP) which is the Locus of the fourth Equation, shall be to the Parameter thereof, as m is to n, because $\frac{n}{m}=\frac{bb}{aa}$. And this is what was propos'd.

*Art. 378.

SCHOLIUM V.

393. HENCE the arbitrary Line a, doing the Office of Unity, is sufficient for constructing the proposed Equation, by means of a Circle and a given Parabola, or else by means of a Circle and an Ellipsis, or Hyperbola similar to a given one. But when it is requir'd to Construct the same by means of a Circle, and an Ellipsis, or Hyperbola given, then one arbitrary Line only is not enough; there must be others brought into the Equation proposed, to the End that they may be determin'd afterwards so as the given Section serves. This we shall show how to do in the following Lemma.

A FUNDAMENTAL LEMMA *for the Construction of Equations of the Third and Fourth Degree, with a Circle and an Ellipsis, or a given Hyperbola.*

394. LET there be an Equation of the fourth Degree, *viz.* $z^4 + abzz - aacz + a^3d = 0$, wherein the Letters, a, b, c, d, do denote given Lines, and the Letter z, expresses the unknown Roots of the Equation. Assume another unknown Quantity $x = \frac{fz}{a}$ (the Letter f denoting a Line taken at pleasure); and in the Room of z, zz, and z^4, substituting their Values $\frac{ax}{f}$, $\frac{aaxx}{ff}$, and $\frac{a^4x^4}{f^4}$ in the foresaid Equation, and the same will be changed into this, *viz.* $x^4 + \frac{bff}{a}xx - \frac{cf^3}{a}x + \frac{df^4}{a} = 0$; also assume a third unknown Quantity y such, that being multiply'd by f, the Product fy may be equal to xx the Square of the second unknown Quantity; and then we shall have the following Equations.

1. $xx - fy = 0$, and substituting fy, and $ffyy$ for xx and x^4, in the Equation $x^4 + \frac{bff}{a}xx$, *&c.* and we shall have a second Equation.

2. $yy + \frac{bf}{a}y - \frac{cf}{a}x + \frac{dff}{a} = 0$, which being added to the first Equation, and then,

3. $yy + \frac{bf}{a}y - fy + xx - \frac{cf}{a}x + \frac{dff}{a} = 0$, and the Locus of

* *Art.* 324, *and* 329. this Equation is * a Circle, when the unknown and indeterminate Quantities x and y make right Angles with one another. Again, multiply the first Equation by the Fraction $\frac{g}{a}$ (g expressing any Line at pleasure) and then we have $\frac{g}{a}xx - \frac{fg}{a}y = 0$; and adding this Equation to the second, and then substracting the same from it, and the two following Equations will be had.

4. $yy + \frac{bf}{a}y - \frac{gf}{a}y + \frac{g}{a}xx - \frac{cf}{a}x + \frac{dff}{a} = 0$, whose Locus is

* *Art.* 324. * an Ellipsis.

* *Art.* 332. 5. $yy + \frac{bf}{a}y + \frac{gf}{a}y - \frac{g}{a}xx - \frac{cf}{a}x + \frac{dff}{a} = 0$, whose Locus is * an Hyperbola or the opposite Sections.

SCHOLIUM.

395. IF the Signs of ſome Terms of the propoſed Equation ſhould be different from what they are here; or if ſome Terms be wanting, notwithſtanding this, the Loci of the five Equations ſhall always be Conick Sections of the ſame Species; that is, the Loci of the firſt and ſecond Equations ſhall be always Parabola's, the Locus of the third, a Circle, *&c.*

PROPOSITION V.

Problem.

396. *TO conſtruct the Equation* $z^4 + abzz - aacz + a^3d = 0$ *of the four Degrees, by means of a Circle given, and an Hyperbola ſimilar to a given Hyperbola; or elſe by means of a given Hyperbola and a Circle.*

Conſtruct * the Loci of the third and fifth Equations, taking the ſame Lines AP, PM, (making a right Angle APM with each other) for the unknown and variable Quantities x and y; and then the Values of the unknown Quantity z may be determin'd by means of the Interſections of theſe Loci, after the following manner. **Art.* 324. *and* 332. FIG. 220. *and* 221.

Through the Point A the Origin of the x's, draw the Line $AD = \frac{af - bf}{2a}$ parallel to PM, and on the ſame Side as PM, when a exceeds b, and on the oppoſite Side when the ſame is leſs. Alſo draw the indefinite right Line DG parallel to AP, and in DG aſſume $DC = \frac{cf}{2a}$ from D towards PM, then about the Centre C with the Radius CF or $CG = \frac{f}{2a}\sqrt{cc + aa - 2ab + bb - 4ad}$ deſcribe a Circle. This being done, draw $AH = \frac{bf + gf}{2a}$ parallel to PM but towards contrary Parts, and draw the indefinite Right Line HK parallel to AP, in which take $HI = \frac{cf}{2g}$ from H the contrary way to PM, and on both Sides the Point I aſſume IK, IL, each equal to $\frac{f}{2g}\sqrt{cc - bg + 4dg}$ or $\frac{f}{2g}\sqrt{bg - 4dg - cc}$ (b being taken $= \frac{\overline{b+g}^2}{a}$ for brevities ſake). Laſtly, with the Axis LK (which muſt be a firſt Axis when $cc + 4dg$ is greater than bg, and a ſecond one when it is leſs) having the ſame proportion to its Parameter KO, as a is to g, deſcribe an Hyperbola or the oppoſite Sections meeting the Circle in the Points M, M, from which Points let MP,

MP,

MP, be drawn perpendicular to the Line AP. I say the Parts AP, AP, of this Line shall be the Roots of the Equation $x^4 + \frac{bff}{a}xx - \frac{cf^3}{a}x + \frac{df^4}{a} = 0$, the affirmative Roots falling from A towards the Line PM which was drawn in the Construction, and the negative ones the contrary way.

For by the Properties of the Circle and Hyperbola, we shall have the third and fifth Equations; and substracting the third Equation from the fifth, and then $\frac{gf}{a}y + fy - \frac{g}{a}xx - xx = 0$, and so $y = \frac{xx}{f}$; and substituting $\frac{xx}{f}$ for y, and $\frac{x^4}{ff}$ for yy, in either of those two Equations, and then there will arise the Equation x^4, &c. But the Values of x being had, the Values of z will be so likewise; because $z = \frac{ax}{f}$.

Now to satisfy the first thing requir'd in the Problem, call the Radius of the given Circle CF, r; and then r shall be $= \frac{f}{2a}\sqrt{cc+aa-2ab+bb-4ad}$; whence if you take $f = \frac{2ar}{\sqrt{cc+aa-2ab+bb-4ad}}$ the Radius CF or CG of the Circle being the Locus of the third Equation, shall be equal to the given Quantity r. And that the Hyperbola be similar to a given one, that is, that its first or second Axis LK, be to the Parameter KO in the given Ratio of m to n; you need only assume $g = \frac{an}{m}$, because $LK : KO :: a : g :: m : n$.

Lastly, To order it so that the Hyperbola be given, or, which is the same thing, that the first or second Axis LK, and KO the Parameter of that Axis, be equal to given Lines; call the first Axis LK, $2t$; and its Parameter KO, p; then $KO\ (p) = \frac{2gt}{a}$, and $LK\ (2t) = \frac{f}{g}\sqrt{cc+4gd-bg}$ (remembring that $b = \frac{\overline{b+g}^2}{a}$); whence $g = \frac{ap}{2t}$, and $f = \frac{2gt}{\sqrt{cc+4dg-bg}}$: And so it appears, if $cc + 4dg$ does exceed bg, and those Values be taken for g and f, the given Lines $2t$ and p shall be found for the first Axis LK, and its Parameter KO, in the Construction of the fifth Equation. But if $cc + 4dg$ be less than bg, then you must call the second Axis LK, $2t$; and its Parameter KO, p;

from whence there arises (as above) $g = \frac{ap}{2t}$, and $f = \frac{2gt}{\sqrt{bg - cc - 4dg}}$. And if bg, in this last Supposition, wherein $2t$ does represent the second Axis, exceeds $cc + 4dg$; then if those Values be taken for g and f, in the Construction of the fifth Equation, the given Lines $2t$, and p, shall be found for the second Axis LK, and its Parameter KO.

Here it must be observ'd, that the Value of f may be imaginary in both these Suppositions; and so it appears, that the Construction in this Case is impossible, at least by this Method. And since all those who have used the same after *Slusius*, who invented it, have affirm'd, that it is general; I shall here orderly examine all the Cases that can happen, and shew, that even in this Example there may be an infinite Number of Cases wherein that Method will not succeed.

If the given Hyperbola's be two Conjugate ones, the Construction will be always possible; for if you call the first Axis of one of the Hyperbola's LK, $2t$; and its Parameter KO, p; the Value of $f = \frac{2gt}{\sqrt{cc + 4dc - bg}}$ will be imaginary, that is, bg does exceed $cc + 4dg$; then you need but use the Hyperbola, that is a Conjugate to this Hyperbola, and its second Axis instead of it, and the first Axis thereof; because the second Axis of the latter Hyperbola being the same as the first Axis of the other, the Value of f will not then include a Contradiction. *Note*, if $cc + 4dg$ be $= bg$, then the Equation of the fourth Degree may be brought down to one of the second.

SCHOLIUM.

397. 1. If the given Hyperbola be an Equilateral one, we have $g = a$, and then the first Axis of that Hyperbola must be used in the Construction of the Problem, when $cc + 4dg$ does exceed hg, that is, substituting $\frac{\overline{b+g}^2}{a}$ for its Value b, and a for g, when $cc + 4ad$ does exceed $\overline{b+a}^2$; and the second Axis, when the same is less. And the Construction shall be always possible.

2. If the first Axis of the given Hyperbola exceeds its Parameter: The first Axis thereof must be used in the Construction of the Problem, when $cc + 4ad$ is greater than $\overline{b+a}^2$; for it follows from thence, that $cc + 4dg$ does exceed bg, that is, (multiplying by $\frac{a}{g}$, and substituting $\frac{\overline{b+g}^2}{a}$ for its Value b) $\frac{acc}{g} + 4ad$ is greater than $\overline{b+g}^2$, since

$g\left(\frac{ap}{2t}\right)$ being less than a in this Supposition, the Quantity $\frac{acc}{g}+4ad$ shall be greater than $cc+4ad$, and $\overline{b+g}^2$ shall be less than $\overline{b+a}^2$. And contrariwise, when $cc+4ad$ is less than $\overline{b+a}^2$; then the second Axis must be used; for it follows from thence, that $cc+4dg$ is less than bg, or $\frac{acc}{g}+4ad$ less than $\overline{b+g}^2$, since $2t$ denoting here the second Axis, which is less than the Parameter p, the Quantity $\frac{ap}{2t}$ is greater than a. Whence it appears, that the Construction is always possible, not only when the Hyperbola given is an Equilateral one, but also when the first Axis is greater than its Parameter.

3. If the first Axis be less than its Parameter. Then there is a Necessity of using the first Axis, when $cc+4ad$ does exceed $\overline{b+a}^2$; for if the second Axis be used, then $cc+4dg$ must be less than bg, or $\frac{acc}{g}+4ad$ less than $\overline{b+g}^2$; which cannot be, because $2t$, which then would express the second Axis being greater than p, the Quantity $g\left(\frac{ap}{2t}\right)$ would be less than a. But in using the first Axis, it may happen that $\frac{acc}{g}+4ad$ be less than $\overline{b+g}^2$, because $g\left(\frac{ap}{2t}\right)$ is greater than a; and then it is evident, that the Construction of the Problem will become impossible, because there is a Contradiction imply'd in the Value of $f\left(\frac{2gt}{\sqrt{cc+4dg-bg}}\right)$. In like manner, when $cc+4ad$ is less than $\overline{b+a}^2$, there is a Necessity of using the second Axis; and because then the Value of $g\left(\frac{ap}{2t}\right)$ is less than a, it may so fall out, that $\frac{acc}{g}+4ad$ be greater than $\overline{b+g}^2$, and so the Value of $f=\frac{2gt}{\sqrt{bg-cc-4dg}}$ may be imaginary.

Therefore it is evident, that there may a Multitude of Cases happen, wherein the Construction of the Equation in the Problem is impossible; and that is, when the first Axis of the given Hyperbola is less than its Parameter, for otherwise the same shall always succeed.

COROL.

COROLLARY I.

399. IF the fourth Equation in the last Problem should have been taken instead of the fifth, and the Locus of that Equation, which is an Ellipsis, been constructed, instead of the Hyperbola being the Locus of the fifth Equation; then it is plain, that the proposed Equation z^4, *&c.* might have been constructed by means of a given Circle, and an Ellipsis similar to one given; or else by means of a given Ellipsis and a Circle.

COROLLARY II.

399. THE foregoing Construction may be made general for all Equations of the third and fourth Degree, after the following manner: 1. Get the second Term out of the given Equation, when the same has one; afterwards multiply the same by its Root z, if it be but of the third Degree; take a Plane ab, equal to all the Planes multiplying zz, a Solid aac equal to all the Solids which multiply z; and finally, a surd Solid a^3d equal to all the given surd Solids. 2. Strike out the Terms wherein b is found in the Values of AD, DC, CF, AH, IH, LK, when zz is not in the given Equation, as also the Terms wherein c or d happen, when the fourth or fifth Term is wanting: And change the Signs of all the Terms wherein b is found of an odd Dimension, if the third Term of the Equation given has a Sign different from the third Term of the aforesaid Equation; also change the Signs wherein c or d are found of an odd Dimension, when the fourth or fifth Terms have Signs different from the fourth or fifth Terms of the precedent Equation. 3. Take those Lines towards PM when their Values are positive, and the contrary way when negative.

SCHOLIUM.

400. THE Construction aforegoing may always be render'd more simple in particular Equations proposed to be constructed, if it be so ordered that a be equal to b; for then the given Equation need only be reduced to this Form, *viz.* $z^4 \mp aazz \mp aacz \mp a^3d = o$, instead of $z^4 \mp abzz \mp aacz \mp a^3 d = o$.

PROPOSITION VI.

Problem.

401. TO *find the Roots of the following Equation,* $z^4 - bz^3 - aczz + aadz + aahh = o$ *by means of a given Hyperbola between its Asymptotes, and a Circle.*

 Make

Fig. 222. Make $z = \frac{ax}{f}$, and then the given Equation may be transform'd into this $x^4 - \frac{bf}{a}x^3 - \frac{cff}{a}xx + \frac{df^3}{a}x + \frac{bbf^4}{aa} = 0$.

From any Point *M* of the given Hyperbola, whose Centre is the Point *A*, draw *MP* parallel to *AQ* one of the Asymptotes, meeting the other Asymptote in the Point *P*; then call *AP*, x; *PM*, y; and supposing mm to be the Power of the Hyperbola, we shall have $xy = mm$.

Now if you take $f = \sqrt{\frac{a}{b}}$, then $bff = amm$, and $\frac{bbf^4}{aa} = m^4 = xxyy$; and substituting $xxyy$ for $\frac{bbf^4}{aa}$, which is the last Term of the given Equation, and dividing by xx, then there will arise $xx - \frac{bf}{a}x - \frac{cff}{a} + \frac{df^3}{ax} + yy = 0$, which will be changed (by substituting in the Term $\frac{df^3}{ax}$ instead of x its Value $\frac{bff}{ay}$ found by means of the Equation $xy = mm = \frac{bff}{a}$) into this, viz. $xx - \frac{bf}{a}x - \frac{cff}{a} + \frac{df}{b}y + yy = 0$,

* Art. 308, 310. the Locus of which is * a Circle, when the Angle *APM* is a right Angle.

But when the Angle *APM* is not a right Angle, (or, which is the same thing; when the given Hyperbola is not an Equilateral one; then it is evident, that the Locus of the last Equation is not a Circle, but an Ellipsis. But to find an Equation, whose Locus is a Circle, in the Asymptote *AP* assume $AB = 2a$; draw *BE* parallel to the other Asymptote *AQ*, and draw *AE* from the Centre *A* perpendicular to *BE*. Then calling the given Quantities *BE*, g; *AE*, e; multiply the Equation $xy - mm = 0$, whose Locus is the given Hyperbola, by $\frac{g}{e}$, and then there will arise $\frac{gxy}{e} - \frac{gmm}{e} = 0$. This being done, add this last Equation to the preceding one, when the Angle form'd by the Asymptotes is acute, and subtract it from the same when that Angle is obtuse, as it is suppos'd to be in the Figure; then we shall have $yy - \frac{g}{e}xy + \frac{df}{b}y + xx - \frac{bf}{a}x - \frac{cff + gmm}{a} = 0$, and the Locus of this

* Art. 317, 319. is * a Circle, which may be thus constructed.

Fig. 222. In the Asymptote *AQ* assume $AD = \frac{df}{2b}$ from *A* towards *PM*: Draw the Line $DC = \frac{bf}{e} - \frac{def}{2ab}$ parallel to *AE*, on the same Side

AP

AP with regard to PM, when the Value of DC is positive; and on the contrary Side, when negative: Then about C as a Centre, with a Radius $CM = \sqrt{\overline{AC}^2 + \frac{cff - gmm}{a}}$ describe a Circle. I say, this Circle shall cut the given Hyperbola, and that opposite to it, in Points (M) from which Parallels MP being drawn to the Asymptote AQ; the Parts AQ of the other Asymptote shall express the Roots of the Equation $x^4 - \frac{bf}{a}x^3 - \frac{cff}{a}xx + \frac{df^3}{a}x + \frac{bbf^4}{aa} = 0$; the affirmative Roots falling from A towards PM, and the negative ones the contrary way.

For from the Nature of the Circle, will be had the following Equation $yy - \frac{g}{a}xy + \frac{df}{b}y + xx - \frac{bf}{a}x - \frac{cff + gmm}{a} = 0$, which will become (putting mm for xy) $xx - \frac{bf}{a}x - \frac{cff}{a} + \frac{af}{b}y + yy = 0$, and substituting $\frac{mm}{x}$, or $\frac{bff}{ax}$ for y, and its Square for yy, in this last Equation, and then there will be had again the proposed Equation $x^4 - \frac{bf}{a}x^3$, &c.

If the Angle form'd by the Asymptotes be acute, then in the Values of AD and CM, the Signs of the Terms where g happens must be changed, because BE (g) will then be negative. But when the Hyperbola is an Equilateral one, the Terms wherein g is found must be struck out, and $2a$ must be put for its Value e; because AE then falls in AB; from whence the Construction is render'd much more simple.

Now the Values of x being found, the Values of z will be also had, since $z = \frac{ax}{f}$. And this is the thing propos'd.

COROLLARY I.

402. If the last Term of a proposed Equation of the fourth Degree has the Sign $-$, it is manifest by working as above, that an Equation will be gotten, wherein is the Term yy with the Sign $-$, and consequently the Locus of the same will not be a Circle, but an * Hyperbola. Whence it appears, that the foregoing Method will not do for Equations of the fourth Degree having the Sign $+$ prefix'd to their last Term. *Art. 332.

COROLLARY II.

403. BY the Method aforesaid may be constructed any given Equation of the third Degree, as $x^3 \mp nxx \mp apx \mp aaq = 0$, with a given Hyperbola between its Asymptotes and a Circle. For multiplying the same by $x + r$, when aaq is affirmative, and by $x - r$ when aaq is negative, it may be always changed into this Equation of the fourth Degree.

$$\begin{array}{l} x^4 \mp nx^3 \mp apxx \mp aaqx + aaqr = 0, \\ \quad\ \mp r \quad\ \mp nr \quad\ \mp apr \end{array}$$

whose last Term shall have always the Sign $+$, and so it will be one of those Equations that may be constructed in manner aforesaid.

But this Construction may be very much shorten'd, if you observe, (1.) To take, for the arbitrary Quantity a representing Unity, the Line m, being the Root of the Power of the given Hyperbola, which is the Locus of $xy = mm = aa$, because m is $= a$. (2.) To make an Advantage of the indeterminate Quantity r for comparing the last Term $aaqr$ with $a^4 = xxyy$; by which we get $r = \frac{aa}{q}$. (3.) That the Circle, whose Intersections do determine the Roots of the Equation, shall necessarily cut the Hyperbola when aaq is negative, and that opposite to it when aaq is positive, in the Point K; from whence drawing KH parallel to the Asymptote AQ, the Part AH of the other Asymptote must be equal to r; since $x = \mp r$ is one Root of the Equation of the fourth Degree. From hence arises the following Construction, which may be easily made general for all Equations of the third Degree. FIG. 223.

Suppose the Angle form'd by the Asymptotes to be an acute Angle, and having taken for the arbitrary Quantity a equal to Unity, the Root of the Power of the given Hyperbola, suppose the given Equation of the third Degree to be thus express'd, $x^3 - nxx - apx - aaq = 0$. In the Asymptote AP assume $AB = 2a$, and draw BE parallel to the other Asymptote AQ; also from the Centre A draw AE perpendicular to BE; then in AQ take $AL = q$ from A towards PM, because aaq is negative, in the given Equation; also draw LK parallel to AP, meeting the Hyperbola in the Point K. This being done, call the given Quantities BE, g; AE, e; LK, r; and in the Asymptote AQ assume $AD = \frac{pr}{2a} - \frac{1}{2}q = \mp d$ for Brevity's Sake, and draw $DC = \frac{an + ar \mp dg}{e}$ parallel to AE, in observing to take or draw those Lines towards the same Parts as PM, when their

their Values are positive, and towards contrary Parts when negative. Lastly, About the Centre C, with the Radius CK, describe a Circle, which shall cut the opposite Hyperbola's in Points (M) from which the right Lines MP being drawn parallel to the Asymptote AQ; the Parts AP of the other Asymptote shall be the Roots of the proposed Equation $x^3 - nxx - apx - aaq = 0$.

For if the right Lines MP, KH, be produced, till they meet the Line DC produced likewise (if necessary) we shall have (by means of the right-angled Triangles CFK, CGM) these two Equations $\overline{GM}^2 + \overline{CG}^2 = \overline{CM}^2$, and $\overline{FK}^2 + \overline{CF}^2 = \overline{CK}^2$; and consequently $\overline{GM}^2 + \overline{CG}^2 = \overline{FK}^2 + \overline{CF}^2$, because the Lines CM, CK, are Radii of the same Circle. But by Construction (supposing here for avoiding the Confusion arising from the Signs + and —, that $\frac{pr}{2a} - \frac{1}{2}q = +d$, that is, that this Value is positive) GM or $PM + PG = y + \frac{g}{2a}x + d$, CG or $DG - DC = \frac{ex}{2a} - \frac{an - ar - dg}{e}$, FK or $KH + HF = q + \frac{g}{2a}r + d$, CF or $CD - DF = \frac{an + ar + dg}{e} - \frac{er}{2a}$. Now, if the Analytick Values of those Lines be substituted in the aforesaid Equation $\overline{GM}^2 + \overline{CG}^2 = \overline{FK}^2 + \overline{CF}^2$ we shall have first $yy + \frac{g}{a}xy + 2dy + \frac{gg+ee}{4aa}xx - nx - rx = qq + \frac{g}{a}rq + 2dg + \frac{gg+ee}{4aa}rr - nr - rr$, because there is no Necessity of writing down the Squares of d and of $\frac{an+ar+dg}{e}$, which mutually do destroy each other. Now since $xy = rq$, from the Nature of the Hyperbola, and $4aa = gg + ee$, because AEB is a right-angled Triangle, the last Equation may be brought to this, *viz.* $yy + 2dy + xx - nx - rx = qq + 2dq - nr$, wherein substituting $\frac{pr}{a} - q$ for $2d$, and $\frac{aa}{x}$ and $\frac{a^4}{xx}$ for y and yy, and there will come out

$$\begin{array}{llll} x^4 - nx^3 & - apxx & - aaqx & + a^4 = 0: \\ \quad - r & + nr & + apr & \end{array}$$

which being divided by $x - r$, and we shall get $x^3 - nxx - apx - aaq = 0$, which is the Equation proposed in the Problem.

The foregoing Construction may be made general, if you, 1. Assume AL in the Asymptote AQ on the contrary Parts as PM, when

aaq

aaq is affirmative in the given Equation; and change the Signs of the Terms when q and r happen in the Values of AD, DC. And 2. Change the Signs of the Term wherein p happens in the Value of AD, when $ap x$ is affirmative in the given Equation, and strike out the same when that Term is wanting: And do the same with regard to the Term wherein m happens in the Value of DC, when mxx is affirmative. 3. You must change the Term wherein g happens in the Value of DC, when the Angle form'd by the Asymptotes is obtuse, and strike it out when this is a right Angle, always observing that e is $=$ $2n$.

SCHOLIUM.

404. Algebra furnishes us with easy ways of transforming any Equation of the fourth Degree into another of the same Degree, wherein the Signs of the Terms may be alternate. And because then the last Term will have always the Sign $+$ prefixed to it, therefore it is plain using this Preparation, when the last Term of an Equation to be constructed hath the Sign $-$ prefixed to it, that the Method laid down in the Problem shall be general for all Equations of the fourth Degree. But because all the real Roots of an Equation are positive when the Signs of the Terms are alternate; therefore there will then be only need of using the given Hyperbola, because the opposite one by means of which the negative Roots are determin'd, does become useless.

PROPOSITION VII.

Problem.

405. *TO construct the following Equation of the sixth Degree* viz. $x^6 - bx^5 + acx^4 + aadx^3 + a^3exx - a^4fx + a^5g = 0$, or $x^6 - bx^5 + cx^4 + dx^3 + exx - fx + g = 0$, *(wherein the Line* a, *which renders the Number of Dimensions in each Term equal, and is esteem'd as Unity, is understood) by means of a Circle and a Locus of the third Degree.*

Take $x^3 - mxx - nx + q = -pxy$ for the Locus of the third Degree, wherein the Quantities m, n, p, q, which are look'd upon as being given, must be determin'd in such a manner as to satisfy the Problem; and this may be done thus: If each Side of this Equation be squared, then

$$\begin{array}{l} x^6 - 2mx^5 + mmx^4 + 2mnx^3 + nnxx - 2nqx + qq = ppxxyy \\ \qquad\quad\;\; - 2n \qquad\; + 2q \qquad\;\; - 2mq, \end{array}$$

and if the Terms $-2mx^5$, $-2nqx$, $+qq$, be compar'd with the correspondent Terms $-bx^5$, $-fx$, $+g$, in the proposed Equation, we shall have $m = \frac{1}{2}b$, $q = \sqrt{g}$, $n = \frac{f}{2\sqrt{g}}$; and consequently $x^6 - 2$

$-2mx^5-2nqx+qq=x^6-bx^5-fx+g$. Now if $ppxxyy-mmx^4$, &c. be substituted for its Value $x^6-2mx^4-2nqx+qq$, found by means of the preceding Equation, and $-cx^4-dx^3$, &c. for its Value x^6-bx^5-fx+g found by means of the given Equation; and if you divide by xx, and bring all the Quantities over to one Side; then the following Equation will be had, *viz.*

$$\begin{array}{llll} ppyy & -mm\,xx & -2mn\,x & -nn=0, \\ & +2n & -2q & +2mq \\ & +c & +d & +e \end{array}$$

whose Locus shall be a Circle * if the Quantity $c+2n-mm$, which does multiply xx, be positive, and pp be $=c+\frac{f}{\sqrt{g}}-\frac{1}{4}bb$; for dividing by pp, and making $2r=\frac{2mn+2q-d}{pp}$ and $ss=\frac{2mq+e-nn}{pp}$ or $\frac{nn-2mq-e}{pp}$, for brevity's sake, then we shall have $yy+xx-2rx\mp ss=0$: The Square ss being affirmative, when $2mq+e$ exceeds nn, and negative when the same is less,

*Art. 328, and 329.

Now the Curve which is the Locus of the Equation $x^3-xx-nx+q=-pxy$ may be drawn after the following Manner. Suppose $AP(x)$, $PM(y)$ to be two unknown and indeterminate right Lines forming a right Angle APM with one another; and through the Point A, the Origin of the x's, draw the indefinite right Line AQ parallel to PM, in which assume $AG=\frac{n}{p}$, from A tending the same way as PM, and $GB=\frac{q}{mp}$, the contrary way, and draw the right Line $BC=m$ perpendicular to AQ. This being done, upon some separate Plane describe a Parabola MEM, with the Line $p=\sqrt{c+2n-mm}$ for the Parameter of the Axis, and place the Plane of this Parabola on the Plane wherein AP, and PM are drawn, so that the Axis of the Parabola coincides with the Line AQ, and the Parabola itself tends the contrary way to that which PM does, and in the Axis assume $EF=BG=\frac{q}{mp}$, (from E the Origin) within the Parabola. Then take a long Rule CF, and place it in the Point C, so as to be moveable about the same, and always pass through the Point F: This being done, move that Rule about the Point C, so as to slide the part EF of the Axis of the Parabola along the Line AQ. I say the two continual Intersections M, M, of that Rule, and the Parabola MEM will by this Motion describe two Curves that shall be the Locus sought. FIG. 224.

For

For by Construction AB or $AG - GB = \frac{n}{p} - \frac{q}{mp}$, and by the Property of the Parabola $EQ = \frac{xx}{p}$ because AP or $MQ = x$. But because the Triangles FQM, MDC, are similar, therefore FQ or $EQ - EF\left(\frac{xx}{p} - \frac{q}{mp}\right) : QM(x) :: DM$ or $PM - AB\left(y + \frac{q}{mp} - \frac{n}{p}\right) : CD\ (m - x)$. Whence multiplying the Means and Extremes, and we shall have $x^3 - mxx - nx + q = -pxy$, and if the Points M be taken successively in the three Angles that follow this here Angle, the same Equation will be found always, observing to take $AP = -x$, and $PM = -y$, when the Points P and M, do fall the contrary way to what they do here: So that the two Curves, which may be called *parabolick Conchoides*, shall be the compleat Locus of all the affirmative and negative Values of the unknown Quantity y, answering to all the affirmative and negative Values of the other unknown Quantity x, in the Equation $x^3 - mxx - nx + q = -pxy$.

But now to construct the Circle being the Locus of the Equation $yy + xx - 2rx \mp ss = o$, in the indefinite right Line AP assume $AH = r$, from A towards PM, when the Value of r is positive, and the contrary way, when the same is negative; then about the Centre H with the Radius $HM = \sqrt{rr \pm ss}$, (*viz.* $-ss$ when ss is positive in the Equation, and $+ss$, when the same is negative) describe a Circle, which will be that sought; for because HPM is a right-angl'd Triangle, therefore we have always $\overline{HM}^2 = \overline{HP}^2 + \overline{PM}^2$, that is, $yy + xx - 2rx \mp ss = o$, by substituting the analytick Values, and bringing all the Terms over to one Side.

Now if from the Points M there be drawn Perpendiculars (MQ) on the indefinite right AQ, I say these Lines shall be the Roots of the Equation proposed; the affirmative Roots being on the right, and the negative ones on the left of AQ; for if MP be drawn parallel to AQ, we shall have the following Equation from the Property of the Conchoides, *viz.* $x^3 - mxx - nx + q = -pxy$, that is (squaring both Sides) $ppxxyy = -x^6 - 2mx^5$, &c. and by the Nature of the Circle, $yy + xx - 2rx \mp ss = o$, which being multiplied by $ppxx$ and then $ppxxyy$ is $= -ppx^4 + 2pprx^3 \pm ppssxx$. And comparing these two Values of $ppxxyy$ together, there will be an Equation form'd, wherein if the Values of $2r$, ss, pp, m, n, q, be substituted, there will be found again the proposed Equation $x^6 - bx^5$, &c.

If dx^3 had been negative in the proposed Equation, then you need but have taken $2r = \frac{2mn + 2q + d}{pp}$, and the rest of the Construction would have been the same as above, because d does not happen in the Value

Value of r. And because then all the Signs of the Terms of the proposed Equation are alternate; it is a received Maxim in Algebra, that all the real Roots thereof are affirmative, that is, if the said Equation has two real Roots, and four imaginary ones, the two real Roots are affirmative; if it has four real Roots, and two imaginary ones, the four real Roots are affirmative; and if all the six Roots be real ones, then they shall be all affirmative. From whence it appears, that in this Case we have Occasion only for the Conchoid, which is describ'd by that half of the Parabola next to the fixed Point C, because the other Conchoid only determines the negative Roots.

If the Value of the Radius should happen to be equal to nothing, or imaginary only; or if it should be so small, as not to touch or cut the two Conchoids; then we may be assured, that all the Roots of the Equation will be imaginary. If it cuts them in six Points, all the Roots will be affirmative. And lastly, if it cuts them in but four or two Points, then there will be only four or two affirmative Roots, and the others are imaginary. Here it must be always observ'd, that if the Circle touches one of the Conchoides, the Point of Contact must be esteem'd as two Points infinitely near to one another, so that then the proposed Equation will have two Roots, each equal to the Perpendicular drawn from the Point of Contact to BE.

SCHOLIUM I.

406. FROM the Description of the two parabolick Conchoids, it follows, (1.) That the right Line BE, both ways indefinitely produced, is a common Asymptote to both those Curves. (2.) That one of the Conchoids does pass through the fixed Point C, and the Rule CF touches it in the Point C; because the Point M coinciding with the Point C, the Rule passes through two Points of the Curve infinitely near to one another. (3.) That when the Point F does fall on B, the Rule CF, whose Intersections (M, M) with the Parabola, do describe the Conchoids, falls in CB; and so the Line MFM going through the Point F, does become a double Ordinate; that is, the Line CB meets the Conchoids in two Points K, L, being such that BK and BL are each equal to the Ordinate to the Axis of the Parabola that passes thro' the Point F. From whence it is evident, if BC was equal to that Ordinate, that the Point K would then fall on C; and so the Line BC, which would pass through two Points K and C, of the Conchoid infinitely near to one another, would touch the Curve in the Point C, wherein K and C do coincide.

Any Number of Points of the Conchoids may be found without using the Parabola MEM, after the following manner. In BE take BQ equal to the Parameter of the Parabola, and with any right FIG. 225.

Line OR greater than OB, as a Diameter, describe a Circle cutting BC in the Points D, D; in that Diameter take $RS = EF$, and through the fixed Point C, draw the two right Lines CM, CM, parallel to DS, DS, which shall meet DM, DM, (parallel to EB) in M, M, which shall be two Points of the Conchoids. For if CM be produced till it meets the Asymptote BE in the Point F, and MQ be drawn parallel to BC; then it is plain, that the right-angl'd Triangles MQF, DBS, shall be equal, and so FQ is $= BS$. But if RS be taken $= EF$, we shall have $EF + FQ$, or $EQ = RS + SB$ or RB; and the Parabola EM, whose Vertex is E, and having a Line equal to BO, for the Parameter, shall pass through the Point M; because by the Nature of the Circle, the Square of BD or MQ is equal to the Rectangle under BD or MQ, and the Parameter BO; which likewise follows from the Property of the Parabola: Therefore the Point M, found by this Construction, is the same as that gotten by the Intersection of the Rule CF, and the Semi-parabola EM.

If the Point D was given, and the Point R was requir'd, you must draw DR perpendicular to OD; and the rest of the Construction will be the same as before.

It is proper to observe here by the way, (1.) If in BC you take BD (from B towards C) equal to the affirmative Root of the Equation of the third Degree $x^3 - \frac{2}{3}mxx - \frac{1}{3}mnp = 0$, (the given Quantities BC being $= m$, $EF = n$, $BO = p$); and then the Point M be found as above: This Point shall be further distant from the right Line BC, than any other Points of the Portion KMC, so that the Tangent passing through that Point shall be parallel to BC. (2.) If BD be taken in BC, produced on the other Side of the Point B, equal to the affirmative Root of the Equation $x^3 - mnp = 0$; then the Point M of the Conchoid answering to the Point D, shall be the Point of Inflexion of the Curve; that is, the Point wherein the Curve, from being concave, begins to become convex. Because this does depend upon the Principles that are laid down in my Book, *Des infiniment petits*, therefore I refer you to that Book, or some other of the like Nature; for the same may be here suppos'd as true, without enquiring into the Reason thereof; since this has nothing to do in the Construction of Equations of the sixth Degree, which is the Business here in hand.

Scholium II.

407. THE Description of the two parabolick Conchoids require, (1.) That the Line $BC (\frac{1}{2} b)$ be of some Magnitude, and so the 2d Term of the proposed Equation must not be wanting. (2.) That the Term q cannot be $= 0$, in the Equation $x^3 - mxx - nx + q = -pxy$, since dividing by x, there will arise $xx - mx - n = -py$, the

the Locus whereof is a Parabola; from whence it is plain, that the last Term g in the propos'd Equation must have the Sign $+$ prefixed to it, for $q = \sqrt{g}$.

Further, if the Term fx should have the Sign $+$, you must give the Sign $-$ to the same, in likewise changing the Signs of the 2d and 4th Term; and doing this will not bring any Inconveniency to the Construction, but only will change the negative Roots into affirmative ones, and the affirmative ones into negative ones. And in order for the Locus of the second Equation to be a Circle, $\frac{f}{\sqrt{g}} \mp c$ must be greater than $\frac{1}{4}bb$, (c being affirmative when cx^4 is so, and negative when that is) from whence it appears, that if the Term fx be wanting, the Term cx^4 must be affirmative, and c must be greater than $\frac{1}{4}bb$; and if the Term cx^4 be wanting, $\frac{f}{\sqrt{g}}$ must be greater than $\frac{1}{4}bb$.

Therefore it is evident, that the proposed Equation of the sixth Degree must necessarily have these Conditions, in order to construct the same immediately by means of parabolick Conchoids, and a Circle, according to the Rules above prescribed.

SCHOLIUM III.

408. WHEN an Equation given is one of the fifth Power, by raising it to the sixth, we can very often bring the same to such Conditions, as to be immediately constructed; as will appear by the following Examples.

1. Let $x^5 - a^4 b = o$, and suppose a to be greater than b. Multiply this Equation by $x - b$, and then the following Equation of the sixth Degree will arise, *viz.* $x^6 - bx^5 - a^4 bx + a^4 bb = o$, which has all the requisite Conditions specified in the last Scholium.

2. Let $x^5 - 5aax^3 + 5a^4 x - a^4 b = o$, and let a be greater than b. Multiply this Equation by $x - b$, and then we shall have $x^6 - bx^5 - 5aax^4 + 5aabx^3 + 5a^4 xx - 6a^4 bx + a^4 bb = o$, which hath all the necessary Conditions.

3. Let $x^5 - ax^4 - 4aax^3 + 3a^3 xx + 3a^4 x - a^5 = o$. Multiply this Equation by $x - 4a$; and then there will arise $x^6 - 5ax^5 + 19a^3 x^3 - 9aaxx - 13a^5 x + 4a^6 = o$, which is an Equation of the sixth Degree, wherein all the Terms do happen to have the necessary Conditions.

Here it may be observed that by Means of the first Equation $a^5 - a^4 b = o$, can be found four mean Proportionals between two Extremes A and B; and by means of the second $x^5 - 5aax^3$, &c. a given Angle may be divided into five equal Parts; and by means of the third $x^5 -$

ax^4, &c. a regular Polygon of Eleven Sides may be inscribed in a given Circle; as will appear in the following Book. I shall now proceed to the Construction of the first of these Equations, that so it may be compar'd with the Construction laid down by *Descartes* at the End of his third Book of Geometry.

FIG. 226. Describe a Parabola ME with a Line $p = \sqrt{aa - \frac{1}{4}bb}$ for the Parameter, and assume the Line $AG = \frac{aa}{2p}$, GB or $EF = 4AG$, $BC = \frac{1}{2}b$, $AH = \frac{5aab}{4pp}$, and a Line $s = \frac{a}{2p}\sqrt{4bb - aa}$, or $\frac{a}{2p}\sqrt{aa - 4bb}$; then describe a parabolick Conchoid (as is directed in Article 404.) by means of the Parabola ME, and long Rule CF, freely turning about the fixed Point C, and always passing through the Point F, while the Part EF of the Axis of the Parabola slides along the Line AQ; after this, about the Centre H, with the Radius $HM = \sqrt{\overline{AH}^2 \mp ss}$ describe a Circle, ss being affirmative, when $4bb$ is greater than aa, and negative when it is less. Now if from the Points O, M, wherein this Circle cuts the Conchoid, there be drawn the Lines OR, MP, perpendicular to the Axis AP; I say the Parts AR, AP, shall be the Roots of the Equation $x^6 - bx^5 - a^4bx + a^4bb = 0$. This is proved as in Art. 404.

The trouble of finding a Line $s = \frac{a}{2p}\sqrt{4bb - aa}$, or $\frac{a}{2p}\sqrt{aa - 4bb}$ may be spared, if you consider that the Circle describ'd with the Centre H, must cut the Conchoid COM in the Point O being such, that OR drawn perpendicular to AP, we have $AR = b$; because one Root of the Equation is $x = b$. From whence in AP, assume $AR = b$, and draw RO perpendicular to AP, meeting the Conchoid COM in O; and then you may describe the Circle about the Centre H, with the Radius HO. For the Circle cuts the Conchoid in another Point M, being such that MP drawn perpendicular to AP, the Line AP shall be the greatest of four mean Proportionals required. Because the Circle described about the Centre H does cut the Conchoid passing thro' the Point C, only in two Points O, and M, and does not meet the other Conchoid; therefore the proposed Equation $x^6 - bx^5$, &c. hath only two affirmative Roots AR, AP, and four imaginary Roots.

SCHOLIUM IV.

409. WHEN the given Equation of the sixth Degree hath not the necessary Conditions for being constructed immediately by the Method above explain'd, or else being an Equation of the fifth Degree, the last Scholium is found useless; then the Preparation assign'd by *Descartes* in the third Book of his Geometry may be used. Wherein is

is shewn how to transform any Equation of the fifth or sixth Degree into another of the sixth, having the Signs of all the Terms alternate, and where the known Quantity in the third Term does exceed the Square of $\frac{1}{2}$ the known Quantity in the second: For by this means the Construction of the Problem is made general for all Equations of the fifth and sixth Degree. The Method of doing of this I shall not here explain, since it does depend upon pure Algebra, which is not my Design here to treat of; and because in the following Proposition I shall lay down a general Way of constructing all Equations of the fifth and sixth Degree, without any other Preparation than only getting out the second Term.

PROPOSITION VIII.

Problem.

410. *To find the Roots of the following Equation* $x^6 - bx^4 - cx^3 + dxx - fx + g = 0$, *by means of a given Cubick Parabola, and a Conick Section.*

Let $aay = x^3$ be an Equation whose Locus is a Cubick Parabola *MAM* (*AP* being $= x$, $PM = y$, $AB = a$). Instead of x^6 in the proposed Equation, substitute its Value a^4yy, in the Room of x^4 its Value $aaxy$, and instead of x^3 its Value aay; then the proposed Equation will be changed into this Equation of the second Degreee, $yy - \frac{b}{aa}xy - \frac{c}{aa}y + \frac{d}{a^4}xx - \frac{f}{a^4}x + \frac{g}{a^4} = o$, whose Locus is * an Ellipsis, when d is greater than $\frac{1}{4}bb$, that is, when the known Quantity which does multiply xx, is greater than the Square of half the known Quantity multiplying x^4, as is here supposed. And if you would have the Line that represents Unity, and which is conceiv'd to be in the proposed Equation, equal to the Parameter a of the given Cubick Parabola; then the Equation will be changed into this $yy - \frac{b}{a}xy - cy + \frac{d}{a}xx - fx + ag = o$ whose Construction is as follows. FIG. 227. *Art. 323.

In the indefinite Line *AP* take $AB = a$, and draw the right Lines $BE = \frac{1}{2}b$, $AD = \frac{1}{2}c$, parallel to *PM* and towards the same Parts: Also through the Point *A* draw the right Line *AE* (e), and through the Point *D*, the right Line *DG* parallel to *AE*, in which take $DC(s) = \frac{2afe + bce}{4ad - bb}$ from *D* towards *PM*, and on both Sides Cassume *CK*, *CL*, each equal to $t = \sqrt{ss + \frac{ccee - 4agee}{4ad - bb}}$. This being done, with the

the Diameter LK ($2t$) having a Line $KH = \frac{4adt - bbt}{2ee}$, and its Ordinates parallel to PM, describe an Ellipsis, and the same shall be that sought.

Now if from the Points of Concurrence of this Ellipsis, and the given Cubick Parabola there be drawn Lines as PM, making with AP, the Angle APM, given or taken at pleasure, then the Parts (AP) of the indefinite right Line whereon the indeterminate Quantity x tends, shall be the Roots sought; the affirmative Roots falling on that Side the Point A as PM is supposed to fall in the Construction, and the negative Roots on the contrary Side, For by the Property of the Conick Section $yy - \frac{b}{a}xy - cy + \frac{d}{a}xx - fx + ag$ is $= 0$, and by the Property of the Cubick Parabola, y is $= \frac{x^3}{aa}$; and substituting that Value instead of y, and the Square thereof for yy, in the precedent Equation, and then we shall get the given Equation $x^6 - abx^4 - aacx^3$, &c. $= 0$.

SCHOLIUM I.

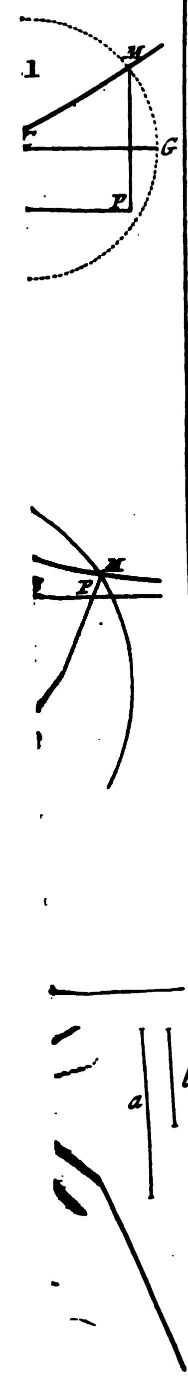
411. ANY Equation of the fifth or sixth Degree being given, if you get out the second Term, and afterwards multiply it by the unknown Quantity x, if the same be an Equation of the fifth Degree, and if by means of (a) the Parameter of the given Cubick Parabola, you reduce the known Quantities that multiply x^4 to the Expression ab; those that multiply x^3, &c. to the Expression aac, then if the Substitution as above be used, the given Equation shall be always transform'd to a Locus of the second Degree. Whence if a Cubick Parabola be once well describ'd, with any Line a for a Parameter, and any Angle (APM) made by the Ordinates (PM) to the Diameter AP, then it is manifest that the Roots of any Equation of the fifth and sixth Degree may be found by means of that Curve, and a proper Conick Section.

SCHOLIUM II.

412. IF after the second Term of a given Equation of the fifth or sixth Degree be gotten out, and the Equation is multiply'd by x, if it be one of the fifth Degree; the known Quantity that multiplies the Square xx be positive, and does exceed the Square of half the known Quantity that multiplies x^4: Then from the Substitution by means of $aay = x^3$, we shall have always an Equation of the second Degree, whose Locus is an Ellipsis, as appears in the Problem. And it may be so ordered always, that that Ellipsis may become a Circle, but then the Cubick Parabola will not be given. For Example.

Find

ge 262.



nd * a Line a whereof the squared Square a^4 be equal to the known *Art. 378.
ntity multiplying xx, and by means of this Line reduce all the n Quantities multiplying x^4 to the Expression ab, all these mul-ing x^3, &c. to the Expression aac; and then the Equation shall duced to this Form $x^6 \pm abx^4 \pm aacx^3 + a^4xx \pm a^4fx \pm a^5g = o$. substituting a^4yy, $aaxy$, and aay for x^6, x^4 and x^3, and this E-ion of the second Degree will be had $yy \pm \frac{b}{a}xy \pm cy + xx \pm fx$ $g = o$, the Locus of which shall be a Circle, * if the Angle AEB *Art. 327, and 329.
ade a right Angle; which may be easily done thus.

the indefinite right Line AP assume $AB = a$; and upon Fig. 228.
Line, as a Diameter, describe the Semi-circle AEB, towards the Parts that PM falls, when $\frac{b}{a}xy$ is negative, and towards con- Parts when the same is positive. In the Diameter of this circle assume the Line $BE = \frac{1}{2}b$, from B to E; and drawing AE the Line PM must be drawn parallel to BE, and the rest of the struction will be the same as for the Ellipsis, which will here be a cle; because the Angle CGM shall be a right Angle; and AEB g a right-angled Triangle, ee is $= aa - \frac{1}{4}bb$, which must ex-s the Ratio of the Diameter LK to the Parameter thereof. This cle KMM is the Locus of this Equation $yy - \frac{b}{a}xy - cy + xx$ $fx - ag = o$, which is the same as that in the Problem, only d $= a$.

Now if the Line AB be divided into many equal Parts AP, AP, and allels PM, PM, &c. be drawn to BE; and if every PM be taken al to a fourth Proportional to its Correspondent AP, and the given e AB: Then if a Curve (MAM) be drawn through all the ints (M) thus found; it is plain, that this Curve will be the Locus the Equation $x^3 = aay$; and consequently by means of the Points Intersection (M, M) of the Cubick Parabola and Circle, may be nd all the Roots (AP, AP) of the proposed Equation.

SCHOLIUM III.

3. BEcause Parabola's of all Degrees have been often spoken of in this Book; and since we have shewn how to use a Cubick Para-bola for the Construction of Equations of the fifth and sixth Degree, will not be foreign to our Purpose to examine the several Sorts of nes these Parabola's may make. For which End, let BC, DE be Fig. 229.
o indefinite straight Lines, cutting one another in the Point A, and there be a Parabola AM, of any Degree (in the Angle BAD) hose Nature is such, that MP being drawn from any Point M there-of,

of, parallel to DE, and meeting BC in the Point P, and the indeterminate Quantities AP, PM, and the given Quantity AB being called x, y, and 1; we have always $x^m = y^n$ (where the Letters m and n do denote the Indices of the Powers of x and y, and may be any positive whole Numbers at pleasure, and m is suppos'd to be greater than n) it is evident, 1. When AP (x) is $=0$, PM (y) is so likewise, and the more AP (x) increases, the more does PM (y)

* Art. 237. also increase. 2. The Subtangent $PT\left(\frac{n}{m}x\right)$ is * always less than AP (x) because n is suppos'd to be less than m. Whence it follows, that the Parabola AM, be it of what Degree soever, shall always pass through the Point A; the same recedes infinitely more and more from the right Line BC esteem'd as being a Diameter thereof; and lastly, the Convex Side of it is next to that Diameter. But because the Curve AM, which falls in the Angle DAB, is only a Portion of the Parabola, we must next examine in which of the Angles DAC, CAE, EAB, that Parabola is continued; and here there are three Cases.

Case 1. When the Index (m) of the Power of x is an even Number, and the Index (n) of the Power of y, an odd Number. The Root (m) of x^m shall be $\mp x$, and the Root (n) of y^n shall be only $+y$; for if m be $=4$ and $n=3$, then it is manifest, that the fourth Power of $\mp x$ is always x^4, but it is not so of the Cube of $\mp y$; since the

Fig. 229. Cube of $+y$ is y^3, and the Cube of $-y$ is $-y^3$. Hence it is evident, that AP (x) may be both positive and negative, but PM (y) is always positive; and consequently the Parabola AM must be continued in the Angle DAC, adjoining to BAD, so that if a right Line be drawn through any Point K of the Line AD parallel to BC, the same shall meet the Parabola MAM in two Points M, M, equally distant from the Point K. And this is the common Parabola, which is the Locus of the Equation $xx=ay$, or $xx=y$, supposing the Parameter $a=1$.

Case 2. When the Indices m and n are odd Numbers: The Root (m) of x^m shall be only $+x$, and the Root n only $+y$; but because the Equation $-x^m = -y^n$ is the same as $x^m = y^n$, and the Root (m) of $-x^m$ is $-x$, and the Root (n) of $-y^n$ is $-y$; therefore AP (x) may be both positive and negative, as also PM (y), observing that

Fig. 230. when AP is positive, PM is so likewise, and contrariwise. Whence it appears, that the Parabola AM, in this Case, must be continued in the Angle CAE vertically opposite to DAB, in a Position altogether the same, as in the Angle DAB, but inverted; so that AP being taken equal to AP, and PM drawn making with AP the Angle APM equal to the Angle APM; the right Line PM does meet

meet the Portion AM falling in the Angle CAE, in the Point M being such, that PM is equal to PM. And such is the principal Cubick Parabola $x^3 = aay$, or $x^3 = y$, supposing $a = 1$.

Case 3. When the Index (m) of the Power of x is an odd Number, and the Index (n) of the Power of y an even Number: The Root (m) of x^m shall be always $+ x$, and the Root (n) of y^n will be $\mp y$. For Example; let AM be a Cubick Parabola, which is the Locus of the Equation $x^3 = ayy$, $x^3 = yy$, it is plain that the Cube Root of x^3 is only $+ x$, and the Cube Root of yy is $\mp y$. Therefore the Parabola AM must be continued in the Angle BAE, adjacent to the Angle BAD; in such manner, that if a right Line be drawn through any Point P of the Line AB, parallel to DE, it shall meet the whole Parabola MAM in two Points M, M, equally distant from the Point P. FIG. 231.

Now the general Equation $x^m = y^n$, always appertains to one of the aforesaid three Cases; for if m and n be suppos'd to be both even Numbers, you must extract the square Root of both Sides of the Equation as often as possible; by which means it may be reduced to an Equation, wherein one of the Indexes shall necessarily be odd. And m may be supposed always to be greater than n; for if it should be less, and (for Example) $aax = y^3$, then comparing the Points of the Parabola AM with the Points of the Line DE, and calling AK, x; KM, y; we shall have this Equation $x^3 = aay$, which shall express likewise the Nature of the same Parabola AM, and wherein the Index of the Power of x is greater than the Index of the Power of y; so that the same Reasoning would hold with regard to the Line DE, as before with regard to the Line BC. From whence it is evident, that all Parabola's of any Degree whatsoever, will always have one of the three preceding Figures. FIG. 232.

PROPOSITION IX.

Problem.

414. *It is proposed to construct the following Equation of the eighth Degree* $x^8 - bx^7 + cx^6 - dx^5 + ex^4 - fx^3 + gxx - hx + l = 0$, *wherein none of the Terms are wanting, by means of two Loci; one of the second, and the other of the fourth Degree.*

Take $xx = ay$ for the Locus of the second Degree, and for x^8, x^7, x^6, x^5, x^4, x^3 and xx substitute their Values a^4y^4, a^3y^3x, a^3y^3, $aayyx$, $aayy$, ayx, ay; then if the given Line a be taken for Unity, we shall have this Equation, $y^4 - \frac{b}{a}xy^3 + cy^3 - dxyy + aeyy - afxy + aagy - aahx + a^3l = 0$, the Locus whereof being one of the fourth Degree, and the most simple possible; because one of the un-

known

known Quantities x ariſes no higher than to the firſt Degree, all the Points thereof may be determin'd in uſing right Lines and Circles only.

Now if the Parabola being the Locus of the firſt Equation $xx = ay$ be conſtructed, and if there be taken any Number of different Magnitudes for y, and the Values of x anſwering to them in the ſecond Equation be determined; then the Locus, which ſhall paſs through the Extremities of all the Values of y, and which will be conſequently the Locus of the ſaid Equation, will determine the ſought Values of the Roots of the given Equation, by means of the Points wherein that Locus cuts the Parabola. This is evident, becauſe ſubſtituting $\frac{xx}{a}$ the Value of y in the ſecond Equation, and the Powers of that Value for the Powers of y, and there will ariſe the given Equation $x^8 - bx^7$ &c. $= 0$.

COROLLARY I.

415. BEcauſe Unity a is arbitrary, it may be ſuppos'd given; and ſo the Parabola which is the Locus of the firſt Equation $xx = ay$, is given. Now by means of this Equation it is evident, that any Equation of the ſeventh or eighth Degree may be always transform'd into an Equation of the fourth Degree, wherein the unknown Quantity x is only of one Dimenſion; therefore any Equation of the ſeventh or eighth Degree having all the Terms, or wanting ſome of them, may be conſtructed always by means of a given Parabola, and a Locus of the fourth Degree, in which one of the unknown Quantities has only one Dimenſion, without any other Preparation than taking the Parameter a of the given Parabola for Unity, that ſo the known Quantities multiplying x^6, may be reduced to the Expreſſion ac, thoſe that multiply x^5 to the Expreſſion aad, &c.

COROLLARY II.

416. IT may be prov'd after the ſame manner, that any Equation of the ninth or tenth Degree may always be conſtructed by means of a given Parabola, and a Locus of the fifth Degree, wherein one of the unknown Quantities hath only one Dimenſion: Alſo Equations of the eleventh and twelfth Degree may be conſtructed by means of a given Parabola, and a Locus of the ſixth Degree; and ſo of others to Infinity.

PROPO-

PROPOSITION X.

Problem.

417. *To construct the following Equation of the ninth Degree* $x^9 - bx^7 + cx^6$, &c. $= 0$, *wanting only the second Term, by means of two Loci, each of the third Degree.*

Take $x^3 = aay$ for one of the Loci of the third Degree, and instead of x^9, x^7, x^6, &c. substitute their Values a^6y^3, a^4xyy, a^4yy, *&c.* then if a be taken for Unity, we shall get the other following Equation of the third Degree, *viz.* $y^3 - \frac{b}{a}xyy + cyy$, &c. $= 0$, wherein the unknown Quantity x can rise no higher than the second Degree, because every where wherein x^3 happens in the proposed Equation, aay is substituted for the same.

And it is plain, that if this Locus be constructed with a Cubick Parabola, which is the Locus of the other Equation $x^3 = aay$; then the Intersections of the two Loci shall determine the Roots of the given Equation.

COROLLARY.

418. ANY Equation of the sixth, eighth, or ninth Degree being given, it is manifest, after the second Term be gotten out, and it is afterwards multiply'd by the Root x, (when the Equation is of the eighth Degree) and by xx (when it is one of the seventh) that it shall be always transformed into a Locus of the third Degree, in substituting, as above, by means of the Equation $x^3 = aay$, whose Locus is a given Cubick Parabola; so that this is a general Way for all Equations of the seventh, eighth, and ninth Degree. After the same manner it will be found, that any Equation of the twelfth Degree, not having the second Term, may be transform'd into a Locus of the fourth Degree by means of the said Equation $x^3 = aay$; as likewise Equations of the tenth and eleventh Degrees, by raising them to the twelfth.

But if an Equation of the sixteenth Degree, only wanting the second Term, be propos'd; then by means of the Locus of the fourth Degree $x^4 = a^3y$, that Equation may be transform'd into one of the fifth Degree. After the same manner you will find, that an Equation of the twentieth Degree may be transform'd into a Locus of the sixth Degree, by means of the said Locus of the fourth Degree $x^4 a^3 y$; as also Equations of the seventeenth, eighteenth, and nineteenth Degrees. Moreover, Equations of the 25th only wanting the second Term, may be transform'd into a Locus of the sixth Degree, by means of the Locus of the fifth Degree $x^5 = a^4y$; as likewise all Equations

of the 21st, 23d and 24th Degrees. And this Enquiry may be continued on further at pleasure.

SCHOLIUM I.

419. HENCE it is necessary to observe, that if an Equation of the sixteenth Degree wants the third and sixth Terms, as well as the second; the Locus of the fifth Degree, which conjointly with that of the fourth Degree $x^4 = a^3 y$ serves to construct the Equation, may be transform'd into one of the fourth Degree; and the same Observations may be made upon Equations of higher Degrees. But although it be true, that an Equation of the sixteenth Degree, only wanting the second Term, can be transform'd only to a Locus of the fifth Degree, if for this Effect the Locus of the 4th Degree $x^4 = a^3 y$ (having but two Terms) be used; yet it must not be generelly concluded from thence, that the most simple Loci for resolving a compleat Equation of the sixteenth Degree, must be one of the fourth and the other of the fifth Degree. For, on the contrary, it appears evident, that if a Locus of the fourth Degree, consisting of several Terms, be used instead of $x^4 = a^3 y$, having but two Terms, that Locus may be so chosen, as to serve to transform the compleat Equation of the sixth Degree to another Locus of the fourth. The Reason of which is this; if two Loci of the fourth Degree be assum'd, in one of which the unknown Quantity x arises to the fourth Degree, and in the other the unknown Quantity y; then it is manifest from the Rules of Algebra, that if the unknown Quantity y be gotten out by means of those two Equations, that an Equation will be had, wherein the unknown Quantity x arises to the sixteenth Degree. And because two Loci of the fourth Degree may both together have more than sixteen Terms, since each of them may have fifteen different ones; therefore those Loci may contain all the known Quantities of the given Equation: This is enough to shew the Possibility of constructing a compleat Equation of the sixteenth Degree by two Lines of the fourth.

Further, it ought to be supposed that the two most simple Loci for constructing a compleat Equation of the 20th, 19th, and 17th Degrees, shall be, the one of the fourth, and the other of the fifth Degrees, because the Product of these two Loci arises to the 20th Degree, and they may contain both together more Terms than the proposed Equation, and so do contain all the known Quantities happening therein. And if the unknown Quantity in a proposed Equation has 21, 22, 23, 24, or 25 Dimensions, then the two most simple Loci that w[illegible]uct them, will be those of five Degrees each. From hence arises

arises the following Rule by which may be found the two most simple Loci that can solve a proposed Equation.

Extract the square Root of the highest Power or Degree of the unknown Quantity. If there be no remainder, then each of the two Loci must have the same Number of Degrees as there are Unities contain'd in that square Root; but if there be a remainder, then the same is equal, less or greater than the square Root, if it be equal or less, the Degree for one of the Loci will be the Root itself; and for the other that Root plus Unity: If the Remainder be greater than the Root, then the Degree of both the Loci shall be the Root plus Unity.

For Example, it is required to find the two most simple Loci that can resolve an Equation of the 37th Degree. Because the square Root of 37 is 6, and the Remainder 1 is less than 6, one of the Loci must be of the sixth, and the other of the seventh Degree, which Loci will do also for Equations of the 38th, 39th, 40th, 41st, and 42d Degrees. Again, if an Equation be of the 43d Degree, because the square Root of 43 is 6, and the Remainder 7 is greater than 6, the two Loci must be each of the 7th Degree: Which Loci also serve for Equations of the 44th, 45th, 46th, 47th, 48th and 49th Degrees.

SCHOLIUM II.

420. IT sometimes happens that a given Equation may be constructed by means of one Curve only put into two different Positions, as will appear in the following Example.

It is requir'd to construct this Equation of the 9th Degree, $x^9 + a^8 x - a^8 b = o$, wanting all the middle Terms, except the last but one. Take the Equation $x^3 = aay$, the Locus whereof is the Cubick Parabola MAM, having the given Line $AB = a$ for a Parameter, and right Lines, as PM (y) making an assum'd Angle APM with the correspondent Parts AP (x) of the Axis or Diameter for Ordinates; then cube both Sides of this last Equation, and we shall have $x^9 = a^6 y^3$, which (by Substitution) shall change the proposed Equation into this here $y^3 = aab - aax$, the Locus whereof may be thus constructed.

In AP produced assume $AC = b$, and through the Point C draw the indefinite right Line CK parallel to PM; then with CK as an Axis, whose Parameter is $CD = a$, and Ordinates right Lines as KM parallel to AP, describe another Cubick Parabola MCM. I say this shall be the Locus requir'd. FIG. 233.

For by Construction MK or $CP = b - x$, and by the Property of the Curve $\overline{CK}^3 = MK \times \overline{CD}^2$, that is, in analytick Terms, $y^3 = aab - aax$. And it is evident, 1. That if right Lines as PM be drawn from Points (M) wherein this latter Cubick Parabola MCM meets the other

MAM;

MAM; then the Parts AP shall express the Roots x of the propos'd Equation $x^9 + a^8x - a^8b = 0$. 2. That the Cubick Parabola's MCM, MAM, are the same in all respects, because their Parameters AB, CD, are equal, and the Angles APM, CKM, that are form'd by the Ordinates, are so likewise.

The Situation of the two Cubick Parabola's MAM, MCM, shews that the proposed Equation $x^9 + a^8x - a^8b = 0$, has but one real Root AP (x), which is always affirmative, and less than AC (b); and so all the other eight Roots are imaginary.

PROPOSITION XI.

Problem.

421. *TO construct an Equation of whatsoever Degree, by means of a straight Line, and a Locus of the same Degree, all the Parts of which Locus may always be determin'd by using straight Lines only.*

The last Term of the proposed Equation must be brought to stand alone on one Side, and the whole Equation must be afterwards divided by the Line representing Unity, repeated as often as necessary, that so each of the Terms may express right Lines only: As suppose this Equation was proposed $x^5 - bx^4 + acx^3 - aadxx + a^3ex - a^4f = 0$, then f must be $= \frac{x^5}{a^4} - \frac{bx^4}{a^4} + \frac{cx^3}{a^3} - \frac{dxx}{aa} + \frac{ex}{a}$.

FIG. 334. Now in the indefinite strait Line AB, whose fixed Origin let be in the Point A, assume any Part AP for the Value of x, and draw a right Line $PM = \frac{x^5}{a^4} - \frac{bx^4}{a^4} + \frac{cx^3}{a^3} - \frac{dxx}{aa} + \frac{ex}{a}$ (which may be done

* Art. 376. * always by the use of strait Lines only) parallel to the Line AC given in position; then the Extremity M of this Line shall be one Point of the Curve $ADEM$, whose Points of Intersection M, M, M, &c. made by the right Line KM, drawn parallel to AB through the Point K, so that AK be $= f$, shall determine the Parts KM, KM, KM, &c. which will be the sought Values of the unknown Quantity x in the given Equation.

For if the right Lines MP, MP, MP, &c. be drawn parallel to AC, and the indeterminate Quantities AP, PM, be called x and y; then by the Property of the Curve, $ADEM$, we shall have this Equation PM (y) $= \frac{x^5}{a^4} - \frac{bx^4}{a^4} + \frac{ca^3}{a^3} - \frac{dxx}{aa} + \frac{ex}{a}$ which is a Locus

'1 Degree; and by the Property of the right Line KM this on $y = f$, and by substituting x for y, and multiplying by will arise the proposed Equation $x^5 - bx^4 + acx^3 - aadx +$ $= 0$.

These

These kind of Constructions may be very useful for finding the Limits of Equations. For Example, let us suppose that we have a way to determine the Parts AF, AG, in the Line AC, being such that the right Lines FD, GE, (parallel to AB) do touch the Curve in the Points D, E; then it is manifest, 1. That if AK (f) be less than AF, and greater than AG, as is suppos'd in this Figure, the propos'd Equation will have three affirmative Roots KM, KM, KM, and two imaginary Roots; because the Figure of the Curve is such, that the Line KM shall meet it in three Points, and not more. 2. If AK (f) be less than AG, the Line KM shall cut the Curve in five Points, that is, the proposed Equation will have five affirmative Roots. 3. If AK (f) be greater than AF, the Equation will have but one affirmative Root, and four imaginary ones. 4. If AK be $= AF$, the Equation shall have three affirmative Roots, two whereof will be equal to one another, *viz.* FD, FD. 5 If AK be $= AG$, the Equation shall have five affirmative Roots, two whereof will be equal, *viz.* GE, GE.

The same Curve $ADEM$ being continued from the Point A, shall determine the Roots of this Equation $x^5 - bx^4 + acx^3 - aadxx + a^3ex + a^4f = 0$, only differing from that above in this, that the last Term has the Sign $+$ prefixed to it; which shews that the Line KM must then be drawn below AB, because the Locus thereof must be $y = -f$.

SCHOLIUM.

422. THE preceeding Construction may be varied divers Ways; for instead of making the last Term equal to all the others, you may do the same of any other Term, or even any two being next to each other; and afterwards divide them so, that when they be made equal to y, the Locus of the Equation may be one of the first Degree. For Example, Let there be an Equation of the 3d Degree, $x^3 - abx - aac = 0$; make $\frac{bx}{a} + c = \frac{x^3}{aa}$, then we shall have the two following Equations $x^3 = aay$, and $y = \frac{bx}{a} + c$, the Loci of which being separately constructed after the following Manner, will determine the Roots of the proposed Equation.

Assume two unknown and indeterminate straight Lines AP (x), PM, (y) making any Angle APM with one another, and describe the principal Cubick Parabola MAM, which let be the Locus of the first Equation $x^3 = aay$. Through the Point A, the Origin of the x', draw a right Line parallel to PM, in which take $AC = b$, $AD = c$, both tending towards the same Parts as PM; also in AP produc'd on this Side A, assume $AB = a$; and through the Point D draw an inde- FIG. 235.

finite

finite right Line parallel to BC: I say, if right Lines, as MP, be drawn parallel to AC from the Points (M), wherein this Parallel meets the principal Cubick Parabola MAM, the Parts AP shall be the Roots of the given Equation $x^3 - abx - aac = 0$.

For if DE be drawn parallel to AP, the similar Triangles BAC DEM, will give this Proportion, $BA\,(a) : AC\,(b) :: DE\,(x) : EM = \frac{bx}{a}$, and consequently $PM\,(y) = \frac{bx}{a} + c$. And since MAM is a Cubick Parabola, we have $x^3 = aay$. If then $\frac{bx}{a} + c$ be substituted for y, there will arise the given Equation $x^3 - abx - aac = o$.

If b in the given Equation had been affirmative, then AC must have been taken tending the contrary way to that which PM tends, as also AD if C had been affirmative. From whence it is manifest, that this Construction is general for any given Equation of the third Degree. For when the second Term is gotten out, the Equation may be always reduced to one of the said Forms.

It is plain that a given Cubick Parabola may be used, since you need only assume the arbitrary Quantity a, (taken for Unity) for the Parameter thereof.

PROPOSITION XII.

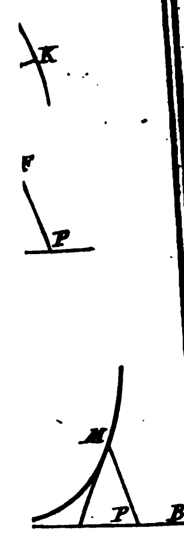
Problem.

423. *TO find the Value of the Roots of any Equation of the third and fourth Degree, or of higher Equations that have but two Terms, to any required Exactness by Approximation; by means of right Lines and Circles only.*

Let a given Equation of the third Degree be $x^3 \mp 2apx - aaq = 0$; multiply this Equation by x, that so it may be rais'd to the fourth Degree, and transposing the Term $aaqx$, and we have $x^4 \mp 2apxx = aaqx$; then add $aapp$ to both Sides, that so one Side of the Equation be a Square, and we get $x^4 \mp 2apxx + aapp = aapp + aaqx$; and extracting the square Root of both Sides, and there comes out $xx \mp ap = a\sqrt{pp + qx}$; lastly, transposing ap, and extracting the square Root again, and then x is $= \sqrt{\pm ap + a\sqrt{pp + qx}}$. This being done, I find that if instead of the exact Value of the affirmative Root x, there be a Magnitude taken greater than it, as c; then it follows, ...at c is greater than $\sqrt{\pm ap + a\sqrt{pp + qc}}$. 2. That $\sqrt{\pm ap + a\sqrt{pp + qc}}$ ...e yet greater than the exact Value of x. This second Proposition ..., but for the first it may be proved thus.

If

5. page 272



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If $2apx$ in the Equation of the third Degree be affirmative, it is manifest that $c^4 + 2apcc >$ * $aagc$; from whence adding the Square $aapp$ to both Sides, and compleating the Calculus as above, there comes out $c > \sqrt{-ap - a\sqrt{pp+qc}}$. But when $2apx$ is negative, by transposing $2apx$ and dividing by x, we shall have $xx = 2ap + \frac{aaq}{x}$, and so if in $\frac{aaq}{x}$ you put for x a Value c greater than the affirmative Root of the Equation $x^3 - 2apx - aaq = o$, the Quantity $2ap + \frac{aaq}{c}$ shall be less than the Square xx (because $\frac{aaq}{c}$ is less than $\frac{aaq}{x}$) and so, much less than the Square cc. Therefore we have $cc > 2ap + \frac{aaq}{c}$, and multiplying by cc, there arises $c^4 - 2apcc > aaqc$, from whence (working as has been directed) and we get $c > \sqrt{ap + a\sqrt{pp+qc}}$. This being suppos'd, I form this Series, $\sqrt{\pm ap + a\sqrt{pp+qc}}$, $\sqrt{\pm ap + a\sqrt{pp+qf}}$, $\sqrt{\pm ap + a\sqrt{pp+qg}}$, &c. wherein f expresses the Term $\sqrt{\pm ap + a\sqrt{pp+qc}}$ immediately going before, and moreover g the Term $\sqrt{\pm ap + a\sqrt{pp+qf}}$, &c.

*This Sign $>$ is one of Majority.

Now from what has been demonstrated, it is evident that all the Terms of this Series will be greater than the exact Value of the affirmative Root x, and that they always approach nearer and nearer to it: So that if the Series be continued on *ad infinitum*, the last Term of the Series shall be exactly equal to the sought Value of the unknown Quantity x. For let z be the last Term, then by the Nature of the Series it is certain, that z shall approach the nearest to the unknown Quantity x of all the other Terms, and consequently the Term $\sqrt{\pm ap + a\sqrt{pp+qz}}$, which would follow it immediately if it was not the last, cannot be less than it; because if it was less, it would approach the nearest to the unknown Quantity x, and consequently would be the last Term, which is contrary to the Hypothesis. And it cannot be greater, because it has been shewn that all the Terms of the Series go on diminishing: Therefore the same shall necessarily be equal to it, and so z is $= \sqrt{\pm ap + a\sqrt{pp+qz}}$, that is, (clearing the Equation of Surds) $z^3 \pm 2apz - aaq = o$, whence it appears that $z = x$. *W.W.D.*

After the same way of reasoning it may be prov'd, that if a Magnitude c be taken less than the exact Value of x, all the Terms of the

Nn said

said Series will go on increasing to the last Term, which shall be exactly equal to the Value of x. We next proceed to shew the manner of constructing the said Series geometrically by right Lines and Circles.

FIG. 236, 237. Draw two indefinite right Lines BD, CP, cutting one another at right Angles in the Point A; and in one of them take $AB=a$, $AD=p$, both on the same Side the Point A, when $2apx$ is affirmative; and one on one Side, and the other on the other, when the same is negative (as is suppos'd in the two Figures) and in the other, take $AC=q$, $AP=c$, always on both Sides the Point A. This being done, on the Diameter CP describe a Semicircle cutting AD in E, and in AC assume AF equal to AE, and in AD from the Point D towards the Point A, in the first Case, and from D the contrary way, in the second, lay off DG equal to DF. Lastly, Upon the Diameter BG describe a Semicircle cutting AP in Q, I say $AQ=\sqrt{ap+a\sqrt{pp+qc}}$. For from the Nature of the Semicircle CEP, the Line AE or AF is $=\sqrt{qc}$; and since FAD is a right-angl'd Triangle, the Hypothenuse FD or $DG=\sqrt{pp+qc}$, and consequently $AG=p+\sqrt{pp+qc}$; and because BQG is a Semicircle, the Line AQ is $=\sqrt{ap+a\sqrt{pp+qc}}$. Now calling AQ, f; and repeating the same Operation, using AQ instead of AP, we shall find $AR=\sqrt{ap+a\sqrt{pp+qf}}$, and afterwards by means of AR, which call g, we shall find $AS=\sqrt{ap+a\sqrt{pp+qg}}$ repeating the same Operation again: So that continuing on the same Operation at pleasure, you will find Lines, as AP, AQ, AR, AS, &c. approaching nearer and nearer to the exact Value of x, the Root of the proposed Equation $x^3-2apx-aaq=0$.

Here you must observe, that $AP\,(c)$ may be taken first of any Length at pleasure; for if this Length be found greater than the Root x, the other Lines AQ, AR, AS, &c. do go on always diminishing; and contrariwise, if AP be less than x, those Lines will go on increasing; so that the true Root is contained between AP of one of the two Figures, and AP of the other, and AQ and AQ, AR and AR, AS and AS. Whence if there be form'd two converging Series, wherein the first Term of one of them is greater than the true Root, and of the other, less; then by taking the correspondent Terms of these two Series, we shall have always the Limits between which the Root must be found; so that the Difference of these Limits diminishes more and more *ad infinitum*.

If the two other Roots of the proposed Equation $x^3-2apx-aaq=0$, had been demanded. Call m the Root found as above by Approximation, and suppose the same to be the true Root; therefore if this be divided by $x-m$, the Remainder will be equal to nothing;

(since the same is $m^3 - 2apm - aaq = o$, and x is suppos'd $= m$) and the Quotient shall be $xx + mx + mm - 2ap = o$, and if this Quotient or Equation be resolv'd, we shall have the two Roots requir'd.

All Equations of the third Degree may be reduced to one or the other of those two Forms; for if the second Term be gotten out, and aaq is affirmative; then putting aaq negative will only change the affirmative Roots into negative ones, and the negative ones into affirmative ones. From whence it appears, that the aforesaid Constructions are sufficient for approximating the Roots of any given Equation of the third Degree. We now pass on to those of the fourth.

It is requir'd to approximate the Roots of the following Equation of the fourth Degree $x^4 - 3apxx - aaqx - a^3r = o$. First find the Roots of this Equation of the third Degree nearly by Approximation, according to the manner before prescrib'd, *viz.*

$$\begin{array}{l} y^3 - 3ppy + 2p^3 = o. \\ \quad\; + 4ary + 8apr \\ \qquad\qquad\;\; - aqq \end{array}$$

wherein you must observe to write $- 2p^3$ when $+ 3apxx$ is in the proposed Equation; $- 4ar$ when $+ a^3r$ is in the same; and lastly, $- 8apr$ when the Signs of the Terms $3apxx$ and a^3r are different. Now one of the approximating Roots y must be supposed to be exact, and having found a Line $v = \sqrt{ay \mp 2ap}$, viz. $+ 2ap$ when $3apxx$ is negative, and $- 2ap$ when the same is affirmative; and then for the four approximating Roots of the proposed Equation, we shall have those of these two Equations of the 2d Degree $xx - vx + \frac{ay \mp ap}{2} - \frac{aaq}{2v} = o$ and $xx + vx + \frac{ay \mp ap}{2} + \frac{aaq}{2v} = o$ (observing to take $- ap$ when $3apxx$ is negative, and $+ ap$ when the same is affirmative, in the proposed Equation) which may be easily constructed by means of strait Lines and Circles. All this is only a Continuation of the Rule laid down by *Descartes*, in the Third Book of his Geometry, for reducing any Equation of the fourth Degree into one of the third, from whence knowing one of the Roots of the proposed Equation, we have all four; and since this depends upon pure Algebra, it may be here supposed as demonstrated; yet the Demonstration being short take it as follows.

The Equation of the fourth Degree $x^4 - 3apxx - aaqx - a^3r = o$, must be esteem'd as being the Product of two Planes $xx - vx + ab - ac = o$, and $xx + vx + ab + ac = o$, wherein the Letters v, a, b, c, do denote unknown Quantities that must be determin'd hereafter so, that the Product of these two Equations which is $x^4 \begin{smallmatrix} -vvxx \\ +2abxx \end{smallmatrix} - 2acvx \begin{smallmatrix} +aabb \\ -aacc \end{smallmatrix} = o$,

 may

may be equal to the proposed Equation. For this End, if the correspondent Terms be compared, we shall have 1. $c = \frac{aq}{2v}$. 2. $b = \frac{vv - 3ap}{2a}$. 3. $bb - cc = -ar$, or $bb - cc + ar = o$; that is (putting for b and c the Values that we have thus found) the Equation $v^6 - 6apv^4 \begin{smallmatrix} +9aappvv \\ +4a^3rvv \end{smallmatrix} - a^4qq = o$. And if you make $vv = ay + 2ap$, then (by Substitution), will be found this Equation of the third Degree

$$y^3 \begin{smallmatrix} -3ppy \\ +4ary \end{smallmatrix} \begin{smallmatrix} +2p^3 \\ +8apr \\ -aqq \end{smallmatrix} = o,$$

from whence knowing one Root y, we shall have the Value of v, by taking the square Root of $ay + 2ap$, and then the Values of b and c, which being put in the two plane Equations first supposed, and there will be found two other plane Equations, whose Product shall be equal to the proposed Equation, and the Resolution thereof shall consequently furnish the four Roots sought. If it be required only to find one affirmative Root of an Equation of the fourth Degree, this may be done immediately by a Series, after the following Manner.

Let $x^4 \mp 2apxx - aaqx - a^3r$ be $= o$, then working after the same manner as for an Equation of the third Degree, and x will be $= \sqrt{\mp ap + a\sqrt{qx + pp + ar}}$ from whence (making $pp + ar = nn$, for brevity's sake) we shall have this converging Series c, $\sqrt{\mp ap + a\sqrt{nn + qc}}$, $\sqrt{\mp ap + a\sqrt{nn + qf}}$, $\sqrt{\mp ap + a\sqrt{nn + qg}}$, &c. whose Construction is the same as that of the Series before given, only AF must be $= n$, and $DG = FE$.

If a^3r had been affirmative, then will x be $= \sqrt{\mp ap + a\sqrt{qx + pp - ar}}$, and when pp exceeds ar, the same converging Series as above will be found (in making $pp - ar = nn$). But it must be observed, when $2apx$ is affirmative in the given Equation, then $\sqrt{qx + pp - ar}$ must be greater than p, that so $\sqrt{-ap + a\sqrt{qx + pp - ar}}$ may not include a Contradiction; from whence x is $> \frac{ar}{q}$ and consequently c must be taken greater than $\frac{ar}{q}$.

If pp be less than ar, then making $ar - pp = qn$, we shall have this converging Series c, $\sqrt{\mp ap + a\sqrt{qc - qn}}$, $\sqrt{\mp ap + a\sqrt{qf - qn}}$, $\sqrt{\mp ap + a\sqrt{qg - qn}}$, &c. wherein it must be observed, that when $2apxx$ in the given Equation is negative, x must be greater than n or $\frac{ar - pp}{q}$

that

that so $\sqrt{ap + a\sqrt{qx+pp-ar}}$ the Value of x may not include a Contradiction, whence c must be taken greater than n.

When r is affirmative in the given Equation, then it may happen that all the four Roots are imaginary, in which Case we shall certainly fall into some Contradiction in the Construction of the Series: For the Demonstration of the converging of the Series depends upon the Supposition of the given Equation's having one affirmative Root. Finally, the Construction of the last Series is something different from the Construction of the others, but because it is not more difficult, I shall omit the same.

This Method becomes troublesome when extended to compleat Equations, exceeding the fourth Degree; for which reason I shall content my self with applying the same to an Equation of the fifth Degree, having only two Terms, from whence may be known how to extend the same to others more compound that have likewise but two Terms.

Let $x^5 - a^4 b$ be $= 0$, then multiplying by x, and transposing, and there will arise $x^6 = a^4 bx$, and extracting the square Root, we have $x^3 = aa\sqrt{bx}$, or $x^4 = aax\sqrt{bx}$, and continually extracting the square Root twice more, x will be $= \sqrt{a\sqrt{x\sqrt{bx}}}$, from whence we get the following converging Series c, $\sqrt{a\sqrt{c\sqrt{bc}}}$, $\sqrt{a\sqrt{f\sqrt{bf}}}$, $\sqrt{a\sqrt{g\sqrt{bg}}}$, &c. whose geometrical Construction is thus.

Draw two indefinite strait Lines BD, CP, cutting one another at right Angles in the Point A, and in one of them assume $AB = a$, and in the other $AC = b$, $AP = c$, on both Sides the Point A. This being done, upon the Diameter PC describe a Semicircle, cutting BA in D, and in AC take $AF = AD$, and then upon the Diameter PF describe another Semicircle cutting AD in E. Lastly, on the Diameter BE describe a Semicircle cutting AP in Q; then it is plain that AQ is $= \sqrt{a\sqrt{c\sqrt{bc}}}$. Now calling AQ, f; and repeating the same Operation, using AQ instead of AP, and you will find $AR = \sqrt{a\sqrt{f\sqrt{bf}}}$, and also $AS = \sqrt{a\sqrt{g\sqrt{bg}}}$. And the right Lines AP, AQ, AR, &c. do come nearer and nearer to the exact Value of the unknown Quantity (x) of the given Equation $x^5 - a^4 b = 0$. This may be prov'd after the same manner as before, for the Equations of the third Degree. FIG. 238.

M. Bernoulli invented these Series, as may be seen in Page 455. of the *Acta Eruditorum*, for the Year 1689.

PROPO-

PROPOSITION XIII.

Problem.

424. *A Portion of a Conick Section being given, to find the Roots of a given Equation of the third and fourth Degree by means thereof.*

It appears in the preceding Problem, that an Equation of the fourth Degree being given, we can by means thereof find always one of the third, from whence knowing one Root, we have all the four of the proposed Equation, by straight Lines and Circles only. Also it is well known, that any Equation of the third Degree may be brought to this Form $x^3 \mp 2apx - aaq = o$, whereof one of the Roots is affirmative, and the two others negative or imaginary. This being laid down; let $x^3 \mp 2apx - aaq = o$ be an Equation of third Degree, whose
FIG. 239. Roots are requir'd to be found by means of the given Portion (BD) of a Parabola, having the Line CH for the Axis, and the Point C the Origin thereof. From the Points B, D, the Extremities of the given Portion, draw the Perpendiculars BG, DH, to the Axis; then it is manifest, that if the affirmative Root was greater than BG, and less than DH, the Circle describ'd about the Centre E, (found as directed at the End of *Art.* 387. for Equations wanting the second Term) with the Radius EC, would certainly cut the Portion BD in some Point M, from whence drawing the Perpendicular MQ to the Axis, this Line MQ would be the affirmative Root thereof. Now when the said Circle does not cut the Portion BD, the Equation must be transform'd into another, whereof the Root may be contain'd between the Limits BG, DH. In order for this, call the given Lines BG, f; DH, g; and suppose we have two Limits m, n, (where m is less than n, and f than g) between which the true Root x is contain'd. Then x will be greater than m, and less than n, and multiplying every Term by f, and dividing by m, and there comes out $\frac{fx}{m}$ greater than f, and less than $\frac{fn}{m}$. Now if z be made $= \frac{fx}{m}$, and instead of x you put its Value $\frac{mz}{f}$ in the Equation $x^3 \mp 2apx - aaq = o$, this Equation will be transform'd into this here $z^3 \mp \frac{2apff}{mm} - \frac{aaqf^3}{m^3} = o$, whose Root $z = \frac{fx}{m}$ shall be greater than f, and less than $\frac{fn}{m}$. Therefore, if the Limits m and n be such, that $\frac{fn}{m}$ be equal to, or less than g, then you

need

need but construct this last Equation by means of *Art.* 387, for having the affirmative Root MQ (z) thereof, by means of the given Portion BC. From hence may be gathered the following Construction.

By the last Problem make two converging Series approaching to the true Root x, of the given Equation $x^3 \pm 2apx - aag = 0$, the first Term of one of them being less than x, and of the other greater. Chuse two correspondent Terms m, n, in those Series being such that $\frac{fn}{m}$ may be equal to or less than g: Which may be done always, because f is less than g, and the Difference between m and n does lessen continually. This being done, transform the given Equation into another, as $z^3 + \frac{2apff}{mm}z - \frac{aaqf^3}{m^3} = 0$; the unknown Quantity of which shall be $z = \frac{fx}{m}$, and constructing this by the Directions at the End of Article 387. the Circle will cut always the given Portion BC in some Point, as M, from which drawing the Line MQ perpendicular to the Axis, this Line shall be the affirmative Root z of the last mention'd Equation; and afterwards making $x = \frac{mz}{f}$, that Line x shall be the affirmative Root of the Equation $x^3 \mp 2apx - aaq = 0$.

If you want the two other Roots of this Equation, when they are not imaginary; the Equation need only be divided by the unknown Quantity x *minus*, that Quantity (which we have found) that will bring down the Equation to the second Degree, the two Roots of which may be found by means of a Circle, according to *Article* 380.

All this is so plain, that there need be no more said about it; only observe, that if the given Portion BD were an Ellipsis or Hyperbola, the 398th or 403d Articles must have been used; and then all the Difficulty would be in transforming the given Equation into another, wherein the affirmative Root should have given Limits; and this might be done as above in the Parabola.

The End of the Ninth Book.

BOOK

BOOK X.

Of Determinate PROBLEMS.

A General Proposition.

425. *A Determinate Geometrical Problem being propos'd, to find the Solution thereof.*

The proposed Problem must be esteem'd first as if it was resolv'd, and such Lines as is judg'd most convenient for the Solution thereof must be drawn. Then name all these Lines, (which commonly do form right-angl'd, or similar Triangles) with Letters of the Alphabet, *viz.* the known Lines with the initial Letters, and the unknown ones with the ultimate ones; and go thro' all the Conditions of the Problem, comparing those Lines to one another in the most simple and natural Order possible; that thereby the same Number of Equations may be form'd, as there are unknown Quantities. Lastly, By common Algebra reduce these different Equations to one only, wherein there is but one unknown Quantity, and, if possible, bring it down to a lower Degree; and solve the same by the Directions in the last Book, and then the Problem will be answered. All this will appear by the following Examples.

EXAMPLE I.

FIG. 240. 426. THE right Line AB being given, to find the Point C, out of that Line, such that the right Lines AC, CB being drawn; (1.) The Sum of their Squares may be to the Triangle ABC in the given Ratio of f to g, (2.) And the Angle ACB contain'd under those Lines may be equal to the given Angle GDK.

Suppose the Point C to be that sought, and draw CH perpendicular to AB, which bisect in the Point E. Now call the given Line AE or EB, a; and the unknown ones BH, x; HC, y; and then $AH = a - x$, and $BH = a + x$. Then because AHC, BHC, are right-angl'd Triangles $\overline{AC}^2$ shall be $= aa - 2ax + xx + yy$, and $\overline{BC}^2 = aa +$

2ax

$2ax+xx+yy$; and consequently $\overline{AC}^2+\overline{BC}^2=2aa+2xx+2yy$. But since the Triangle $ACB=AE\times CH\,(ay)$; therefore by the first Condition of the Problem it follows that $2aa+2xx+2yy : ay :: f : g$; from whence multiplying the Extremes and Means, and dividing by $2g$, we get this Equation $aa+xx+yy=\frac{af}{2g}y=2my$, in taking a Line $m=\frac{af}{4g}$ that so the Equation may be cleared of Fractions.

The next thing is to accomplish the second Condition, *viz.* That the Angle ACB be equal to the given Angle GDG. To do which, from the Point G taken at pleasure in the right Line GD, draw the right Line GF perpendicular to the Side DK, produced, if necessary, and from the Point A the Perpendicular AL to the Side BC (also produced, if necessary) that so we may have two right-angl'd similar Triangles ACL, GDF, whereof GDF is given. This being done, call the given Lines DF, b; FG, c; and make $BC=n$, for Brevity's Sake; then because the right-angl'd Triangles BCH, BAL, are similar, therefore $BC\,(n) : CH\,(y) :: BA\,(2a) : AL=\frac{2ay}{n}$. And $BC\,(n) : BH\,(a+x) :: BA\,(2a) : BL=\frac{2aa+2ax}{n}$. And consequently CL or $BL-BC=\frac{2aa+2ax-nn}{n}$. Whence because the Angle ACL must be equal to the Angle GDF, therefore the following Quantities must be Proportionals, *viz.* $CL\left(\frac{2aa+2ax-nn}{n}\right) : AL\left(\frac{2ay}{n}\right) :: DF\,(b) : FG\,(c)$; and so multiplying the Means and Extremes, and then $2aac+2acx-cnn=2aby$ that is (substituting $aa+2ax+xx+yy$ for nn) $aac-cxx-cyy=2aby$, which includes the second Condition of the Problem. Now because we have found the same Number of Equations as there are unknown Quantities, and since all the Conditions of the Problem are taken in; the next thing to be done must be to reduce these Equations by the common Rules of Algebra to one Equation only, wherein there may be but one of the unknown Quantities x or y. This may be done thus: One of the Equations is $aa+xx+yy=2my$, and the other $aac-cxx-cyy=2aby$, or $aa-xx-yy=\frac{2aby}{c}$; which being added together, and we get $2aa=\frac{2aby}{c}+2my$, and so y is $=\frac{aa}{m+f}$ taking $f=\frac{ab}{c}$. And substituting this Value for y, and its Square for yy in the said Equations, and then xx will be $=$

$\frac{aamm-aaff-a^4}{mm+2mf+ff}$ and $x=\frac{a\sqrt{mm-ff-aa}}{m+f}$; from whence it appears, if mm be less than $aa+ff$, the Problem shall be impossible. Now the Construction of the Problem is after the following manner.

FIG. 241. Through E the Middle of AB draw the indefinite right Line ON perpendicular to AB, and through the Point A draw the Line AM making with AB the Angle EAM equal to the Angle DGF, which is given. About the Point M, wherein AM meets the Perpendicular ON, as a Centre, with the Radius MA, describe the Arc of a Circle ACB. Then in EM produced towards M, assume $MN=m$; and join NA, and draw AO perpendicular thereto, meeting NO in the Point O, through which draw a right Line parallel to AB. I say, this Parallel shall meet the Arc of the Circle ACB in the sought Point C.

For drawing CH perpendicular to AB, then will $CH=EO$ be $=\frac{aa}{m+f}$, because the right-angl'd Triangles NEA, AEO, being similar, we have this Proportion, *viz.* $NE\ (m+f):AE\ (a)::AE\ (a):EO=\frac{aa}{m+f}$. And (by the Nature of the Circle) $\overline{CM}^2=\overline{AM}^2=aa+ff$; and so since MO is $=f+\frac{aa}{m+f}$, and the Triangle MCO is right-angl'd, therefore $\overline{CO}^2$ or $\overline{EH}^2\ (xx)$ is $=aa+ff-ff-\frac{2aaf}{m+f}-\frac{a^4}{mm+2mf+ff}=\frac{aamm-aaff-a^4}{mm+2mf+ff}$. Therefore, &c.

SCHOLIUM.

427. If after all the Conditions of a Problem be taken in, there arises two Equations, in each of which are contain'd both the unknown Quantities; there is no Necessity of reducing both these Equations to one, wherein is only one unknown Quantity, as is prescrib'd in the general Proposition; but the Problem may be resolv'd by constructing separately the Loci of these two Equations, for by means of the Intersections of the two Loci shall the Values of the two unknown Quantities be found. This appears evident in this Example, wherein the right Lines $EH\ (x)$, $HC\ (y)$ forming the right Angle EHC, are taken for the unknown Quantities; and by the Conditions of the Problem are gotten these two Equations, $aa+xx+yy=2my$, and $aa-xx-yy=2fy$; for the Intersections of the Circles which are the Loci thereof, will be the Points sought. Now these Circles may be describ'd thus.

The

The circular Arc ACB being describ'd as in the former Construction, about the Centre A with a Radius $AP = m$, describe an Arc, cutting the Perpendicular EM in P. In this Perpendicular assume $EQ = m$ from E towards the Arc ACB, and about the Centre Q with the Radius $QC = EP$, describe a Circle which shall cut the Arc ACB in Points (C), which will be those requir'd.

For from the Nature of this last Circle $\overline{QC}^2$ or $\overline{EP}^2$ $(mm - aa)$ is $= \overline{QO}^2$ $(mm - 2my + yy) + \overline{OC}^2$ (xx) that is, the first Equation $aa + xx + yy = 2my$; and from the Nature of the other Circle ACB, we get $\overline{MC}^2$ or $\overline{MA}^2$ $(ff + aa) = \overline{MO}^2$ $(ff + 2fy + yy) + \overline{OC}^2$ (xx), that is, the second Equation $aa - xx - yy = 2fy$. Therefore the sought Point C will be in both these two Arcs, that is, it will coincide with the Point of Intersection of them.

Hence it appears that there are two different Points (C) that will answer the Problem, when the two Circles cut one another, as in the Square: When they touch one another there will be but one; and when they neither cut or touch, then the Problem will be impossible.

In solving a Problem by two Loci, care must be taken that one does not fall into a more compound Construction, than that of one Equation having only one unknown Quantity x therein. As in this Example, suppose it be requir'd to solve a Problem, (which is the third following) whose Conditions are contain'd in these two Equations, $y = \frac{cd - cx}{b}$, and $\frac{bb}{ff} yy = aa + xx$: If the Loci of these two Equations be used for determining the Problem, then it is plain that a strait Line must be drawn * for the Locus of the first Equation, and *Art. 306.
an Hyperbola * for the Locus of the second, that so the Values of the *Art. 330,
unknown Quantities x and y, may be determin'd by their Intersections. and 332.
But because in bringing these two Equations into one, this Equation of the second Degree will be had, viz. $xx - \frac{2ccd}{cc + ff} x + \frac{ccdd - aaff}{cc - ff} = 0$, which may be constructed by straight Lines and Circles only; therefore it will be a considerable Fault to use an Hyperbola.

EXAMPLE II.

428. THE Square $ABCD$ being given; it is requir'd to draw the right Line AE from the Angle A thereof, so that FE the Part of this Line contain'd between the Side BC, and the Side DC (continu'd out) be equal to a given Line b. FIG. 242.

Let us ſuppoſe the Point E taken upon the Side DC (produced) be ſuch that FE the Part of the Line AE be equal to a given Line b, that is, let us ſuppoſe the Queſtion to be ſolved, and call the given Lines AB, or AD, or DC, or CB, a; and the unknown Line DE, x. This being done, becauſe the Triangles EDA, ECF, are ſimilar, therefore $ED(x) : DA(a) :: EC(x-a) : CF = \frac{ax-aa}{x}$, and ſince ECF is a right angl'd Triangle, therefore $\overline{FE}^2 = \overline{EC}^2 + \overline{CF}^2 = xx - 2ax + aa + \frac{aaxx - 2a^3x + a^4}{xx}$. But becauſe (by the Conditions of the Problem) EF, muſt be equal to b, therefore $xx - 2ax + aa + \frac{aaxx - 2a^3x + a^4}{xx} = bb$, or $x^4 - 2a^3x + 2aaxx - bbxx - 2a^3x + a^4 = 0$. Whence if this Equation be ſolved, the Value of DE (x) ſhall be ſuch, that if the right Line AE be drawn, the Part (FE) thereof comprehended between the Side of the Square CB, and the Side CD continued out, will be equal to a given Line b.

Becauſe the Equation here found is one of the fourth Degree, a Conick Section muſt be uſed in the Conſtruction thereof. Therefore it will be firſt neceſſary to try whether the Equation cannot be brought down to a lower Degree, by the Rules of Algebra; for this End we find that if cc be taken $= aa + bb$, the ſame ſhall be the Product of theſe two Equations of the ſecond Degree $xx + aa - ax - cx = 0$, and $xx + aa - ax + cx = 0$; ſo that the four Roots of the Equation of the fourth Degree $x^4 - 2ax^3$, &c. may be had by finding the Roots of each of theſe two Equations, I ſhall not here ſhew how to find the Roots of the Equation $xx + aa - ax + cx = 0$; becauſe c being greater than a, the Diſpoſition of the Signs ſhews that the Roots are both negative: But the Roots of the other Equation $xx + aa - ax - cx = 0$ being both affirmative, may be determin'd thus.

In the Side AB produced, aſſume $BG = c$, and upon the Diameter AG deſcribe a Semicircle cutting the Side DC (produced) in E. I ſay this Point ſhall be that ſought.

For calling DE, x; and drawing the Perpendicular EH, then we ſhall have $HG = a + c - x$, and by the Property of the Circle $AH \times HG$ $(ax + cx - xx) = \overline{EH}^2 = aa$.

SCHOLIUM I.

429. IF after having brought a Problem to an Equation, the ſame is a compound one having ſeveral real Roots, then it is plain that there is but one of thoſe Roots that expreſſes the Value of the unknown Quantity ſought: But it muſt be well obſerv'd, that all the other Roots may ſerve in ſomewiſe likewiſe to ſolve the Problem; neither

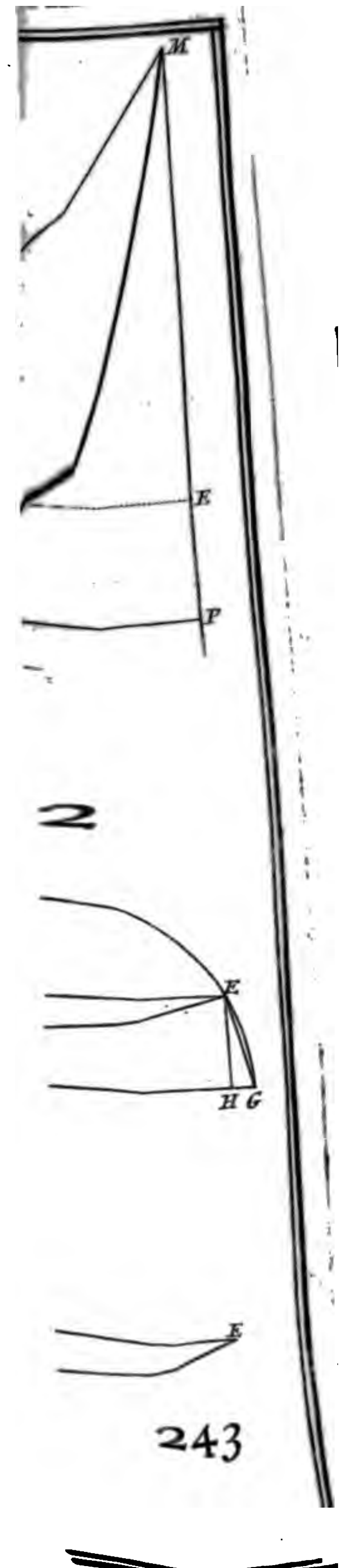
neither will they be different from the Root suppos'd, but only in some particular Circumstances. So in this Example the little affirmative Root DL (x) of the Equation $xx-ax-cx+aa=0$, gives the Point L, upon the Side DC, such that the right Line AL being drawn, meeting the Side BC in K, the Part AL thereof is equal to the given Line b. Moreover, if you take $Bg=c$ in the Side BA produced towards A, then if a Semicircle be describ'd upon the Diameter Ag, this will cut the Side CD produced towards D in the Points e, l, so that De, Dl, shall be the two negative Roots of the Equation $xx+cx-ax+aa=0$: And if the right Lines Ae, Al, be drawn meeting the Side CB produced in the Points f, k, then each of the right Lines ef, lk, shall be yet equal to the given Line b. From whence it appears, that notwithstanding the Value of DE only was what we requir'd in the Problem, yet we have found an Equation whose Roots have furnish'd us with other Values DL, De, Dl, all of which will in some wise answer the Problem.

SCHOLIUM II.

430. IF there be a Suspicion that an Equation taking in the Conditions of a Problem may be brought down to a lower Degree, it will be proper to try other ways different from these we have followed, even when they appear less natural; because it may often happen, that they lead us to more simple Equations; and otherwise, since it is difficult to bring down Compound Equations to lower Degrees. We shall now proceed to solve the aforesaid Problem two other ways, that so what we have here said may be understood.

Suppose the Problem to be resolved, and draw EG perpendicular to AE meeting the Side AB produced in G, and let $AF=y$ and $BG=z$, be the unknown Quantities. This being done, because the right-angl'd Triangles ABF, AEG, are similar, therefore AB (a) : AF (y) :: AE ($y+b$) : AG ($a+z$). And therefore $yy+by=aa+az$. And since there are two unknown Quantities, and the Problem is determinate, there must be another Equation sought. To find which, we must consider that EG is $=AF$ (y); for drawing EH perpendicular to AG, the right-angl'd Triangle EHG is similar and equal to the right-angl'd Triangle ABF, because the Homologous Sides AB, EH, are equal to one another. Therefore (because AEG is a right-angl'd Triangle) $aa+2az+zz$ is $=yy+2by+yy=2yy+2by+bb$, in which Equation substituting, for $2yy+2by$, its Value $2aa+2az$, found by means of the first Equation, and there will arise $aa+2az+zz=2aa+2az+bb$, which may be reduced to this very simple Equation, *viz.* $z=\sqrt{aa+bb}$, which first furnishes the same Construction as above. FIG. 242.

Ano-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Let us suppose the thing done, and upon the Diameter DE describe a Semicircle passing through the Point O, because the Angle DOE is a right Angle; and upon DE, as a Chord, describe the Segment of a Circle containing an Angle equal to the given Angle, whose Arc shall necessarily pass through the Point C. From the Point H, the Centre of a Circle, whereof this Segment is a Part, as likewise from the Points O, C, draw the right Lines HK, OA, CB, perpendicular to DE, and call the given Lines OA, a; CB, b; AB, c; and the unknown ones AK, x; KH, y. Now by the Elements of Geometry it is manifest, 1. That the Point K is the Middle of the Line DE, and consequently is the Centre of the Semicircle DOE. 2. That if the Line PQ be drawn from P, the Centre of the given Angle TPS perpendicular to one of the Sides PT, the Angle QPS form'd by that Perpendicular and the other Side PS, shall be equal to the Angle KEH. And because the Triangles KAO, HKE are right-angl'd, therefore the Square $\overline{KO}^2$ or $\overline{KE}^2$ is $= aa + xx$, and $\overline{HE}^2 = aa + xx + yy$: But producing HK until it meets the right Line CR drawn parallel to DE in the Point R, then because CRH is a right-angl'd Triangle, the Square $\overline{CH}^2$ will be $= bb + 2by + yy + cc + 2cx + xx$. Therefore, since the Lines HE, HC, are Radii of the same Circle, by making an Equality between their Values, and we shall have this Equation, $aa + xx + yy = bb + 2by + yy + cc + 2cx + xx$, then striking out $yy + xx$ from both Sides, and making $\frac{aa - bb - cc}{2c} = d$, (for Brevity's Sake) and y will $= \frac{cd - cx}{b}$.

If Regard be had to the Way we have taken for getting the aforesaid Equation, it will appear that the Circles describ'd about the Centres K, H, with the Radii KO, HC, do meet each other upon the Line DE in the same Points D, E; so that what remains to be done, is only to order it so, that the Angle KEH be equal to the Angle QPS. Whence

In the Line PQ assume PQ equal to CB, and draw QS parallel to the Side PT, and bounded by the other Side PS; then it is evident that the right-angl'd Triangle EKH ought to be similar to the right-angl'd Triangle PQS, and so calling the given Line QS, f, and we shall get this Proportion, EK $(\sqrt{aa+xx}) : KH (y) :: PQ (b) : QS (f)$, therefore y is $= \frac{f}{b}\sqrt{aa+xx} = \frac{cd - cx}{b}$. Now squaring both Sides to get out the Surds, and ordering the Equation, and then we shall have this Equation, $xx - \frac{2ccd}{cc - ff}x + \frac{ccdd - aaff}{cc - ff} = 0$, the Value of one Root of which will determine $AK(x)$; so that if a Circle

Circle be defcrib'd about the Centre K, with the Radius KO, it will cut the Line DE in the two Points fought D, E.

The Roots of the faid Equation may be found by the 380th or 382d Articles: But though the Methods laid down in them are very fimple, confidering their being general, yet there may be very often more eafy Conftructions found by a due Confideration of the Nature of a Problem. For Example: It may be here obferv'd, 1. That if through F, the Middle of the Line OC joining the two given Points, there be drawn the Perpendicular FG, meeting the Line DE given in Pofition in the Point G, then will AG be $= d$; for calling AG, z; the right-angl'd Triangles GAO, GBC, will give $\overline{GO}^2 = zz + aa$, and $\overline{GC}^2 = zz + 2cz + cc + bb$, and making an Equality between thefe two Values, becaufe they are equal, fince the Point G is in the Perpendicular FG, which does bifect the Line OC, therefore $zz + aa$ is $= zz + 2cz + cc + bb$, and fo AG (z) is $= \frac{aa - bb - cc}{2c} = d$. 2. That the Equation $f\sqrt{aa+xx} = cd - cx$ taking in the Conditions of the Problem, may be reduced to this Proportion, $GK\ (d-x) : KO\ (\sqrt{aa+xx}) :: QS\ (f) : AB\ (c)$: So that if the Locus of all the Points K be defcrib'd * fuch, that the right Lines KG, KO, being drawn to the given Points G, O, may be always to one another in the given Ratio of QS to AB; then fhall this Locus cut the Line DE in the Point K fought. From whence may be gotten the following very eafy Conftruction.

* Art. 350.

Through F the Middle of the Line OC, which joins the two given Points, draw the Perpendicular FG, meeting the Line DE given in Pofition in the Point M, fo that $GM : MO :: QS : AB$. And produce GM to the Point N, fo that $GN : NO :: QS : AB$. This being done, with MN as a Diameter defcribe a Circle, which fhall cut the Line DE in the Point K, about which (as a Centre) if a Circle be defcrib'd, this Circle will meet the Line DE in the fought Points D, E.

Becaufe the Circle, whofe Diameter is MN, does cut the Line DE not only in the Point K, but likewife in the Point L; therefore the fame Ufe may be made of the Point L, as of K, for finding two other Points upon the Line DE, which will anfwer the Problem: Whence it appears, that the Problem may have two Solutions.

If the Angle DCE be a right Angle as well as DOE, then it is manifeft that $QS\ (f)$ will become equal to nothing, and fo the Equation $f\sqrt{aa+xx} = cd - cx$ will be chang'd into this here, $cd - cx = o$, and fo x is $= d$, that is, the Centre K falls then in the Point G. And if the Point B be fuppos'd to fall in the Point A, the Equation

$f\sqrt{}$

$f\sqrt{aa+xx}=cd-cx$ will become this $f\sqrt{aa+xx}=\frac{aa-bb}{2}$, by substituting $\frac{aa-bb-cc}{2}$ for cd its Value, and afterwards ſtriking out all the Terms affected with c (which in this Caſe become equal to nothing) and ſo in this Caſe, if an Arc of a Circle be deſcrib'd about the Point O, as a Centre, with the Radius $OK=\frac{aa-bb}{2f}$, it ſhall cut the Line DE in the ſought Point K. This agrees exactly with the 66th, 67th, and 68th Articles, and the general Conſtruction may ſerve without any more to do in an Ellipſis, two of whoſe Conjugate Diameters are given, to find two other Conjugate Diameters that may make a given Angle with each other; and this is what was referr'd in the 65th Article to this Place.

EXAMPLE IV.

432. THREE Points A, B, C, being given, to find ſome fourth Point M; from which, if the three right Lines MA, MB, MC, be drawn to thoſe three Points, the Differences between one of them and each of the two others ſhall be given. FIG. 245.

This Problem is capable of three Caſes. For either all the three Lines MA, MB, MC, are equal between themſelves, or only two of them are equal; or elſe they are all three unequal.

Caſe 1. When the three Lines MA, MB, MC, are all equal to one another; or, which is all one, when the two Differences are nothing; then it is manifeſt, that the Point M ſought ſhall be the Centre of a Circle paſſing through the given Points A, B, C.

Caſe 2. When two of the given Lines MA, MB, MC, as MA, MB, muſt be equal to one another; or (which is the ſame thing) when one of the Differences is nothing, then through the given Point C draw the right Line CO perpendicular to the Line AB joining the two other given Points A, B; and from the Point M (ſuppos'd to be that ſought) draw the right Lines MP, MQ, parallel to CO, OB; then it is plain that AP ſhall be equal to PB, becauſe AM muſt be equal to MB. Now calling the given Quantities AP or PB, a; OP, b; Oc, c; $AM-MC$, f; and the unknown Quantities AM, PM, z, y; then the right-angl'd Triangles APM, MQC, will give theſe two Equations, $zz=aa+yy$, and $zz-2fz+ff=cc-2cy+yy+bb$; and orderly ſubſtracting each Side of the latter Equation from each Side of the former, and then will $2fz-ff=aa-cc+2cy-bb$, and from hence we get this Proportion, $z:y+\frac{aa-bb-cc+ff}{2c}::c:f$; and ſo the following Conſtruction. FIG. 246.

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Through P the Middle of the Line AB draw the Perpendicular $PD = \frac{aa-bb-cc+ff}{2c}$. Divide the Hypothenuſe AD, produced towards that Part as is neceſſary, in the Points E, F; ſo that $AE : ED :: c : f$, and $AF : FD :: c : f$. Then if a Circle be deſcrib'd with EF for a Diameter, the ſame ſhall cut the Line PD in the Point M ſought.

*Art. 350. For drawing the right Line MA, then it is plain * (by the Nature of the Circle EMF) that $AM\ (z) : MD\ (y + \frac{aa-bb-cc+ff}{2c}) :: c : f$; and zz is $= aa + yy$, ſince PM is perpendicular to AB. And becauſe theſe two Equations take in the Conditions of the Problem; therefore, &c.

If through the other Point N, wherein the Line DP meets the Circumference, there be drawn the right Lines NA, NB, NC; then ſhall NA, NB, be equal to one another, and the Difference between each of theſe Lines and the third Line NC, ſhall be equal to the given Quantity f; ſo that the Point N ſhall alſo anſwer the Problem, but with this Difference, that NC is the greateſt of the three right Lines NA, NB, NC, whereas MC is the leaſt of the three Lines MA, MB, MC.

Fig. 247. This ſecond Caſe may be reſolv'd yet without any Calculus, thus. Suppoſe (as before) that M be the Point ſought, and draw the right Lines MA, MB, MC; and about the Centre C, with the Radius $CD = MA - MC$, deſcribe a Circle $DEKFH$. From the Point D, wherein the Line MC meets this Circle, draw the right Lines DA, DB, to the two given Points A, B, meeting the Circle in the Points E, F, through which Points draw the Radii EC, CF, and the Chord EF. This being done, becauſe $MC + CD$, or MD is $= MA$, and ſince the Lines CD, CE, are Radii of the ſame Circle; therefore the Triangles DMA, DCE, ſhall be Iſoſcelles; and conſequently ſimilar, ſince the Angle at D is common; therefore the Lines CE, MA, ſhall be parallel. After the ſame manner we prove, that the Lines CF, MB, ſhall be alſo parallel; and ſo $DA : DE :: DM : DC :: DB : DF$. Hence it appears, that the whole Difficulty is brought to this, *viz.* to find the Point D in the Circumference of the Circle $DEKFH$ being ſuch, that if the right Lines DA, DB be drawn meeting the Circumference in the Points E, F; the Chord EF may be parallel to the Line AB. And this may be done thus.

From the Point C deſcribe a Circle, having a Line $CD = AM - MC$ for a Radius, and draw AC meeting this Circle in the Point K, H: in the Line AB aſſume AG a fourth proportional to AB, AH, AK; and from the Point G draw the Tangent GE to the Circle $EDHFK$. Then if the right Line AE be drawn through the Point of Contact E, meeting the Circle in the Point D, and the Line DC be

be drawn, and if the Point M be so taken upon DC, that $DM:DC::DA:DE$. I say the Point M will be that sought.

For by the Nature of the Circle $DEKFH$, the Rectangle $HA \times AK$ is $= DA \times AE$: And consequently $BA:AD::AE:AG$: therefore the Triangles DAB, GAE, having the Angle A common, and the Sides about this Angle reciprocally proportional, shall be similar; And so the Angle AEG shall be equal to the Angle ABD; but this Angle AEG being form'd by the Tangent EG, and the Chord DE produced, is measur'd by half the Arc DE. Whence, drawing the Chord EF through the Point F wherein the Line BD meets the Circumference, that Angle shall be equal to DFE; and consequently the Lines FE, AB, shall be parallel to one another. But by Construction $DC:DM::DE:EA::DF:FB$. Therefore the Triangles DMA, DMB, shall be Isoscelles; because the Triangles DCE, DCF, being similar to them, are Isoscelles. Whence the Lines AM, MB, shall be each equal to DM, and so equal to one another; and besides AM, or DM shall be greater than MC by the given Length CD. And this is what was proposed.

Case 3. When the three Lines MA, MB, MC, are unequal. From the given Point C draw the Line CO perpendicular to the Line AB, which does join the two other given Points; and from the Point M, which suppose to be that requir'd, draw the right Lines MP, MQ, perpendicular to the Lines AB, CO. This being done, call the given Quantities AO, a; OB, b; CO, c; $AM - MB$, d; $AM - MC$, f; and the unknown Quantities OP, x; PM, y; AM, z; then will AP be $= a + x$, $BP = b - x$, $CQ = c - y$, $BM = z - d$, $CM = z - f$. Now by means of the right-angl'd Triangles APM, BPM, CQM, we get the three following Equations, 1. $zz = aa + 2ax + xx + yy$, 2. $zz - 2dz + dd = bb - 2bx + xx + yy$. 3. $zz - 2fz + ff = cc - 2cy + yy + xx$; and orderly substracting each Side of the two latter Equations from each Side of the first Equation, and we shall get a fourth and fifth Equation $2dz - dd = aa - bb + 2ax + 2bx$, and $2fz - ff = aa - cc + 2ax + 2cy$. Now instead of yy in the first Equation, substitute the Square of the Value of y found by means of the fifth Equation; and then instead of x and xx their Values found by means of the fourth Equation: By doing thus we shall get an Equation, having only the unknown Quantity z therein arising no higher than a Square. And so the same may be resolved by straight Lines and Circles, according to the Directions in the 380th or 382d Articles. But when the Value of z is found, it will be afterwards very easy to find the sought Point M; for the same will be the Intersection of two circular Arcs, whereof one shall have the Point A for the Centre, and the Line AM (z) for a Radius; and the other the Point B for the Centre, and the Line BM $(z-d)$ for a Radius. FIG. 248.

MR always in the given Ratio of b to $\frac{ad}{c}$. This being done, draw AE perpendicular to HM, and having produced the same to G, so that EG be equal to AE; by the second Case, find the Point M being such, that if the right Lines MA, MG, MC, be drawn; MA, MG, may be equal to one another, and the Difference between MA, MC, may be equal to the given Quantity $2d$. I say the Point M will be that sought.

For by the Property of the right Line HM, we have always this Proportion, MO or $DP\ (x) : MR$ or $QK\left(y + \frac{df}{b}\right) :: b : \frac{df}{b}$ and consequently the Point M must needs be in the Line HM. It will be equally distant from the Points A, G; but besides the Difference between AM, MC, must be the given Difference $2d$. Therefore, &c.

SCHOLIUM.

433. IF the two Sums, made by adding one of the three right Lines MA, MB, MC, to each of the other two, had been given, instead of the two Differences, or else if the Sum of two of them, together with the Difference between one of them, and the third had been given: Then the Point M might have been found by the same Methods as easy as when the Differences are given. But I shall leave this to the Industry of the Learner.

COROLLARY I.

434. HENCE arises the manner of describing a Circle, that shall touch three other given Circles.

For let the Points A, B, C, be the Centres of the three given Circles, and the Point M the Centre of the sought Circle, which touches the given Circles in the Points D, E, F, as *per* Figure. Let the Radii of the given Circles AD be $=a$, $BE=b$, $CF=c$; and the Radius of the Circle sought MD or ME, or $MF=z$. Then will AM be $=z+a$, $MB=z+b$, $MC=z-c$; and therefore $AM-MB=a-b$, $MB-MC=b+c$, $AM-MC=a+c$. Whence it appears that the Question is brought to this, *viz.* to find a Point M, from which if the three right Lines MA, MB, MC, are drawn to the three given Points A, B, C, they may have given Differences. FIG. 250.

COROLLARY II.

435. HENCE we have a way of describing a Conick Section (having the given Point F for a Focus) through two given Points B, C, which shall touch a right Line DE given in Position. FIG. 251, 252.

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(after the ſame manner as the Point A was) by means of theſe two Lines, from which if two right Lines be drawn to terminate in the Focus ſought, they may be equal to one another; and moreover, the Difference between either of them, and the Line drawn from the Point thro' which the Section muſt paſs to the ſought Focus, or elſe the Sum of either of them, and this laſt mention'd Line, ſhall be always given: So that the Problem may be ſolv'd always by the precedent Example, and the Scholium belonging thereto. Laſtly, If it be requir'd to deſcribe a Conick Section that ſhall touch three right Lines given in Poſition, and have a given Point for the Focus thereof; then you muſt find three Points by means of theſe three Lines, as the Point A in the laſt two Caſes was determin'd by help of the Line DE, and the Centre of the Circle paſſing thro' theſe three Points ſhall be the other Focus of the Section, whoſe firſt Axis will be a Line equal in Length to the Radius of this Circle. In all the aforeſaid Caſes it muſt be obſerv'd, that if the ſought Point M be infinitely diſtant from the Point F, then the Section ſhall be a Parabola, whoſe Diameters ſhall be parallel to Lines, which being infinitely continued do terminate in the Point ſought.

EXAMPLE V.

436. A Parabola NCS being given, together with the Arc MN thereof; to find another Arc RS of the ſame, which ſhall be to the Arc MN, in the given Ratio of Number to Number. FIG. 253.

Produce the Axis of the Parabola beyond (C) its Origin, ſo that CA be equal to ½ the Parameter, and deſcribe an equilateral Hyperbola EAF, with the Point C as a Centre, and the Line CA as a Semi-firſt Axis; then draw the right Lines MB, NE, RD, SF, parallel to the Axis CA, meeting the ſecond Axis in the Points H, L, K, O, and the Hyperbola in the Points B, E, D, F, from which draw the right Lines BP, EQ, DG, FI, perpendicular to the Aſymptotes. This being done, it is manifeſt * that the Rectangle $AC \times MN$, or * the Hyperbolick Trapezium $HBEL$ is equal to the Hyperbolick Sector CBE *plus* the Triangle CLE *minus* the Triangle CHB; and alſo $AC \times RS = CDF + COF - CKD$. And ſuppoſing the given Ratio of the Arc MN to the Arc RS be as m is to n (where the Letters m and n denote any whole Numbers at pleaſure); then, by the Conditions of the Problem, we ſhall have this Proportion $AC \times MN$, or $CBE + CLE - CHB : AC \times RS$, or $CDF + COF - CKD :: m : n$, and conſequently $nCBE + nCLE - nCHB = mCDF + mCOF - mCKD$. Now if you make the given Quantities $CP = b$; * Art. 246.

$CQ = c$, the unknown Quantity $CG = x$, and take $CI = x\sqrt[m]{\frac{c^n}{b^n}}$, then

will

ſhall the Arc (RS) of the Parabola intercepted between the Parallels (DR, FS) to the Axis, be to the Arc MN in the given Ratio of n to m.

Here it is neceſſary to obſerve, (1.) That the ſecond Term of the laſt Equation is always negative, becauſe CQ (c) is greater than CP (b); and ſo both the Roots thereof will be affirmative or imaginary, according as the known Quantity in the ſecond Term is greater, equal to, or leſs than $aa\sqrt[m]{\frac{b^n}{c^n}}$ the ſquare Root of the laſt Term. (2.) That when $\overline{CG}^2$ (xx) is one Root of the Equation, then will $\overline{IF}^2$ be the other, becauſe CI being $= x\sqrt[m]{\frac{c^n}{b^n}}$, therefore * $IF = \frac{aa}{x}\sqrt[m]{\frac{b_n}{c^n}}$. But the laſt Term of an Equation, is the Product of its Roots. Therefore * *Art.* 101.
if $a^4\sqrt[m]{\frac{b^{2n}}{c^{2n}}}$ the laſt Term of the preceding Equation be divided by $\overline{CG}^2$ (xx), which is ſuppos'd to be one of its Roots, then will the other Root be $\frac{a^4}{xx}\sqrt[m]{\frac{b^{2n}}{c^{2n}}}$, which is the Square of FI. Whence if CG, CT be taken in the Aſymptotes equal to the Roots of the Equation; and if the Parallels GD, TF be drawn to the Aſymptotes, as alſo the right Lines DR, ES, through the Points D, F, (wherein GD, TF, meet the Hyperbola) parallel to the Axis, then ſhall theſe latter Parallels intercept the Arc (RS) of the Parabola ſought.

If m be $= n$, then the general Equation ſhall become this $x^4 - \frac{bbcc - a^4}{cc}xx + \frac{a^4bb}{cc} = 0$, whoſe two Roots will give us CG (x) $= b = CP$, and CT (x) $= \frac{aa}{c} = QE$; whence it follows, that the Arc RS is found by means of them, ſimilarly ſituate on the other Side the Axis with regard to the Arc MN. And ſince it otherwiſe appears, * that the two Arcs RS, MN, being alike ſituate on each Side the Axis, are equal to one another; therefore that ſerves to confirm the Reaſonings here laid down. Hence, if an Arc (MN) of a Parabola be given, it may be eaſily concluded, that no other Arc (RS) thereof being nearer, or farther from C the Origin of the Axis, and equal to MN, can be found, without ſuppoſing the Quadrature of ſome Hyperbolick Sector, or (which amounts to the ſame thing) the Rectification of ſome Parabolick Arc. * *Art.* 86.

If m be $= 1$ and $n = 2$, then will $x^4 - \frac{2a^4bb - 2ccb^4}{c^4 + bbcc}xx + \frac{a^4b^4}{c^4} = 0$; and if m be $= 2$ and $n = 3$, or, which is the ſame thing, if the Arc

$\frac{ffx}{m-x}$, & multiplying by $m-x$, & transposing the Term $\frac{ffx}{m-x}$, then will $mx - xx + 2fz = \frac{mxx - 2gmx - ffx + mnn}{m-x} = \frac{mxx - mmx - nnx + mnn}{m-x}$ (because the Triangle DEB being right-angl'd, ff is $= bb + gg - 2gm + mm = nn - 2gm + mm$): that is, because the two Parts of the Equation may be exactly divided by $m-x$, $mx - xx + 2fz = -mx + nn$, or $2fz = xx - 2mx + nn$. Lastly, Squaring both Sides, and substituting $xx + aa$ for zz, and this Equation of the fourth Degree will be had,

$$\begin{array}{l} x^4 - 4mx^3 + 4mmxx - 4mnnx + n^4 \\ \qquad\qquad\quad - 4ff \qquad\qquad\quad - 4aaff \\ \qquad\qquad\quad + 2nn \end{array} = 0$$

whose Roots, which may be found by means of a Circle and a given Parabola, or other Conick Section, will give us the Values of AP (x) being such, that if PM be drawn perpendicular to AP, and PD join'd, and if the Angle PDM be made equal to the Angle DPM, the Point M wherein DM, PM, the Sides of the Isoscelles Triangle DPM do cut one another, will be the Centre of the Circle sought, the Radius thereof being MP or MD. Or else if $AP = x$ be taken in the Side AB, and $AQ = \sqrt{xx + aa}$ in the other Side AC, and the Lines PM, QN, be drawn perpendicular to the said Sides; the Point M wherein these Perpendiculars meet, shall be the Centre of the Circle required.

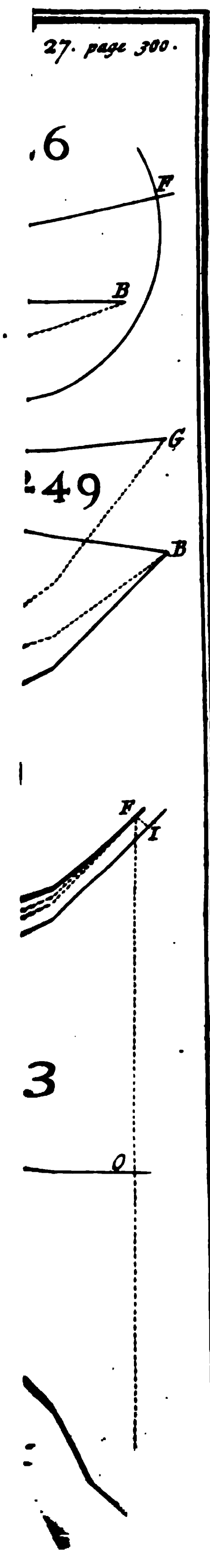
Because nothing is more proper to clear the Understanding, than shewing the different ways by which the Knowledge of the same Truth may be acquired: I therefore shall here resolve this Problem after another way, which in my Opinion is more natural than the former.

Let us suppose the Point M to be the Centre of the Circle sought. Draw the Lines MP, MQ, perpendicular, and the Lines MF, MG, parallel to the Sides of the given Angle BAC, and from the given Point D, draw the right Lines DB, DE, DK, parallel to MP, MF, AB. Now call the given Quantities DB, b; BE, c; DE, f; AB, g; AE, m; AD, n; and the unknown Quantities AP, x; PM or MD, y; then will PB or $DK = g - x$, $MK = y - b$: Whence (because MKD is a right-angl'd Triangle) yy is $= gg - 2gx + xx + yy - 2by + bb$, and consequently $y = \frac{xx - 2gx + bb + gg}{2b} = \frac{xx - 2gx + nn}{2b}$ substituting nn for it Value, $bb + gg$. And because the Triangles DBE, MPF, are similar, therefore DB (b) : BE (c) :: PM (y) : $PF = \frac{cy}{b}$, and so AF or $MG = \frac{bx - cy}{b}$; and because the

Triangles DBE, MQG, are similar, we shall have this proportion $DE\,(f):BD\,(b)::MG\left(\frac{bx-cy}{b}\right):MQ=\frac{bx-cy}{f}$; whence because the Conditions of the Problem require that QC the half of the intercepted Part OC be equal to the given Line a, and that the right Lines MC and MP be Radii of the Circle sought; therefore $\overline{MC}^2=\frac{bbxx-2bcxy+ccyy}{ff}+aa=\overline{MP}^2(yy)$, now multiplying by ff, and we shall have $bbxx-2bcxy+ccyy+aaff=ffyy=bbyy+ccyy$, by putting $bb+cc$ for its Value ff, that is, $ffxx+aaff=ccxx+2bcxy+bbyy$, by striking out $ccyy$ from both Sides, and substituting $ffxx-ccxx$ for $bbxx$; whence, by the Extraction of the square Root, $f\sqrt{xx+aa}$ is $=cx+by=\frac{2cx-2gx+xx+nn}{2}$, by substituting $\frac{xx-2gx+nn}{2b}$ for its Value y, and lastly, putting m for $g-c$, and then will $2f\sqrt{xx+aa}=xx-2mx+nn$. Which is the same Equation as above.

Now we can solve this Problem after a new way which first gives a very easy Construction, but it requires the Description of two Parabola's. 1. I seek the Locus of Points as M, being such, that if a right Line MD be drawn from every of them to the given Point D, and the right Line MP perpendicular to the Line AB given in Position; these two Lines MD, MP, may be always equal to one another. But this Locus is * a Parabola, the Point D being the Focus, and the Line AB the Directrix. 2. I find the Locus of Points as M, being such, that if a Circle be described about every of them as Centres, and passing through the given Point D; this Circle shall intercept the Part OC, of the Line AL given in position, equal to a given Line $2a$. For this End, draw the right Line DL from the given Point D perpendicular to AL, and the right Lines MR, MQ, from one of the sought Points M, (supposed to be given) perpendicular to DL, AL: Then if the unknown and indeterminate Quantities DR, RM (forming a right Angle DRM) be called x, y; the Square $\overline{MD}^2$ is $=xx+yy$, and $\overline{MC}^2=\overline{MQ}^2\,(bb-2bx+xx)+\overline{QC}^2\,(aa)$ because MRD, MQC, are right-angl'd Triangles. But the Lines MD, MC, being Radii of the same Circle, are equal to one another, and consequently $xx+yy$ is $=bb-2bx+xx+aa$, or $yy=bb+aa-2bx$. Therefore if the Parabola being the Locus of this Equation be drawn, it is plain that this Parabola shall pass through M the Centre of the Circle sought. But the Parabola whose Focus is D, and the Line AB also does pass through this Centre; whence the

* Art. 1.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he Centre of the Circle sought shall be in the Intersection of the two Parabola's.

EXAMPLE VII.

.38. A Circle having the Point A for the Centre, and the right Line $A\,M$ for a Radius, being given, together with two Points E, F, in the Plane of that Circle; to find the Point M in the Circumference within the Angle $E\,A\,F$, being such, that if the right Lines $A\,M$, $E\,M$, $F\,M$, be drawn; the Angles $A\,M\,E$, $A\,M\,F$ may be equal to one another. Fig. 255.

If the Lines $A\,E$, $A\,F$, should be equal, then it is manifest that that Line which should bisect the Angle $E\,A\,F$, would cut the Circumference in the Point requir'd. Therefore we suppose those two Lines to be unequal, and also (for avoiding Confusion) that the Line $A\,E$ is less than $A\,F$. And this being premised, we shall solve the Problem two different ways.

Suppose the Point M to be that requir'd, and draw the right Lines $M\,B$, $M\,D$, making the Angles $M\,B\,A$, $M\,D\,A$, with $A\,F$, $A\,E$, equal to the Angles $A\,M\,F$, $A\,M\,E$, and consequently equal to one another; then since the Triangles $A\,F\,M$, $A\,M\,B$, and $A\,E\,M$, $A\,M\,D$, are similar, therefore $A\,F : A\,M :: A\,M : A\,B$. And $A\,E : A\,M :: A\,M : A\,D$. Whence because the Lines $A\,F$, $A\,E$, are given, together with the Radius $A\,M$; the Parts ($A\,B$, $A\,D$) of the right Lines $A\,F$, $A\,E$, shall be given likewise. Now if the right Lines $M\,P$, $M\,Q$, be drawn parallel to $A\,E$, $A\,F$; the Triangles $B\,P\,M$, $D\,Q\,M$ shall be similar, because the Angles $A\,P\,M$, $A\,Q\,M$, are equal; and also the Angles $P\,B\,M$, $Q\,D\,M$, since these are the Complements of the equal Angles $M\,B\,A$, $M\,D\,A$, to two right Angles; and therefore calling the given Quantities $A\,B$, a; $A\,D$, b; and the unknown ones $A\,P$ or $Q\,M$, x; $P\,M$ or $A\,Q$, y; then will $B\,P\,(x-a) : P\,M\,(y) :: D\,Q\,(y-b) : Q\,M\,(x)$: And so (by multiplying the Means and Extremes) we shall get this Equation $xx-ax=yy-by$, or $yy-by-xx+ax=0$, whose Locus being * an Hyperbola may be thus constructed. *Art. 336.

In the Lines $A\,F$, $A\,E$, assume $A\,B$, $A\,D$; third Proportionals to $A\,F$, $A\,M$, and to $A\,E$, $A\,M$, and through (C) the middle of $B\,D$ draw the indefinite right Line $C\,H$ parallel to $A\,B$, in which take $CK=\sqrt{\frac{1}{4}bb-\frac{1}{4}aa}$ (the Line $A\,D$ (b) shall be greater then $B\,A$ (a) because $A\,E$ is supposed to be less than $A\,F$) and then describe an equilateral Hyperbola, having the Point C as a Centre, and the right Line $C\,K$ for a Semi-second Diameter, whose Ordinates ($H\,M$) are parallel to $A\,D$. I say this Hyperbola shall meet the Circumference of the given Circle in the sought Point M.

 For.

For drawing CL parallel to AD, then shall the right Lines CH, CL, bisect the right Lines AD, AB, in the Points O, L; because the Point C does bisect the Line BD, and so CH or $AP - AL$ is $= x - \frac{1}{2}a$, HM or $PM - AO = y - \frac{1}{2}b$. But by the Nature of the equilateral Hyperbola, $\overline{HM}^2$ is $= \overline{CH}^2 + \overline{CK}^2$, that is $yy - by + \frac{1}{4}bb = xx - ax + \frac{1}{4}aa + \frac{1}{4}bb - \frac{1}{4}aa$, whence we get this Equation, $yy - by = xx - ax$, which being turned into a Proportion, and then $BP\ (x-a) : PM\ (y) :: DQ\ (y-b) : QM\ (x)$. Whence because the Angles BPM, DQM, are equal, and the Sides about these Angles are proportional; the Triangles BPM, DQM, shall be similar, and consequently the Angle MBP shall be equal to the Angle MDQ, and their Complements to two right Angles ABM, ADM, shall be equal. But because $AB : AM :: AM : AF$, and $AD : AM :: AM : AE$, the Triangles ABM, AMF, and ADM, AME shall be similar. Therefore the Angle ABM shall be equal to the Angle AMF, and the Angle ADM to the Angle AME; and consequently the Angles AMF, AME, shall be equal to one another, because we have prov'd the Angles ABM, ADM, to be equal.

After the same Manner we prove that the Hyperbola opposite to this, shall cut the Circumference (within the Angle, vertical to EAF,) in the Point M being such, that if the right Lines AM, ME, MF, be drawn, the Angles AME, AMF, shall be equal to one another: As likewise that these two opposite Hyperbola's shall cut the Circumference (within the Angles adjoining to those vertical Angles) each in one Point M being such, that if the right Lines MA, ME, MF, be drawn; the Angle AME shall be equal to the Complement of the Angle AMF, to two right Angles.

* Def. 16. III. *Art. 114. If CG be taken in CL equal to CK, then CG shall be * equal to $\frac{1}{2}$ of the first Diameter, being the conjugate to CK, and so * one of the Asymptotes of the Hyperbola's shall be parallel to KG. But in the Isosceles Triangle GCK, the external Angle GCO (or BAD equal to it) is equal to the two internal and opposite Angles, that is, equal to twice the Angle CGK. Therefore, because the Lines CG, AD, are parallel, the Line KG, and consequently one of the Asymptotes shall be parallel to the Lines bisecting the Angle DAE. Moreover, it is evident that the Line AD is a double Ordinate to the second Diameter CK, because $\overline{OD}^2$ or $\overline{OA}^2$ $(\frac{1}{4}bb) = \overline{CO}^2$ $(\frac{1}{4}aa) + \overline{CK}^2$ $(\frac{1}{4}bb - \frac{1}{4}aa)$; and so one of the opposite equilateral Hyperbola's does pass through the Point D, and the other through the Point A. Now these two Observations do open the way to the following Construction, which is more easy than the precedent one.

In the Lines AF, AE, take AB, AD, third Proportionals to AF, AM, and to AE, AM; and through C the Middle of the Line BD draw

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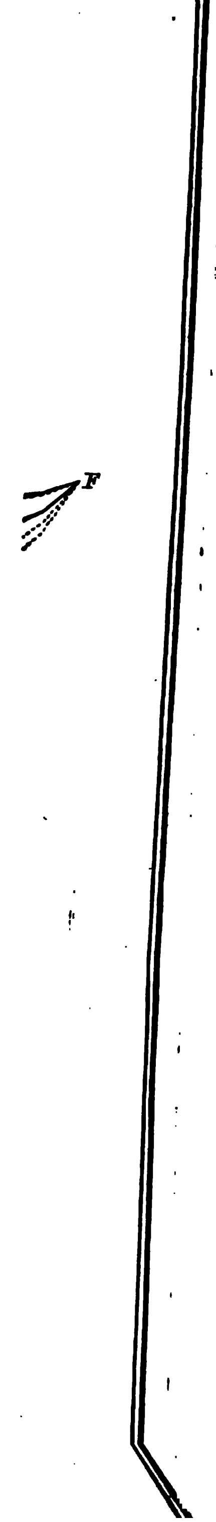

two indefinite Lines *CH*, *CK*, the one parallel, and the other perpen- ... to the Line *AP*, which does bisect the given Angle *EAF*. Then ...n these two Lines, as Asymptotes, describe two opposite Hy- ...a's through the Points *D*, *A*, which shall cut the Circumference ... given Circle in Points, as *M*, being such that if the right Lines ...*ME*, *MF*, be drawn; the two Angles *AME*, *AMF*, shall be ...o one another, when the Point of Intersection *M* does fall in ...ngle *EAF*, or in the opposite Angle to it; and the Angle *AME* ...o the Complement of the Angle *AMF* to two right Angles, ...the Point *M* does fall in either of the Angles adjoining to *EAF* ...opposite Angle.

...s latter Construction may be easily and speedily demonstrated ... Property of the Equilateral Hyperbola, to be found in the 361st ...e, which Property may be otherwise easily prov'd. For if ...he Point *M*, wherein the Equilateral Hyperbola *DM* does meet ...rcumference of the given Circle, there be drawn the right Lines ... *DM* to *B* and *D*, the Extremities of the first Diameter *BD* ...ng the Asymptote *CH* in the Points *O*, *L*, and the Line *AP*, pa- ... to it in the Points *S*, *R*; then by that Article it is evident, that ...s equal to *ML*, and so the Angle *MOL* or *MSR* or *BSA* is ... to the Angle *MLO* or *DRA*. But the Angle *BAS* is equal to ...*R* by Construction, because the Line *AP* does bisect the Angle ...*F*. Therefore the remaining Angles *ABM*, *ADM*, in the two ...ngles *ABS*, *ADR*, shall be equal to one another; whence the An- ...*AMF*, *AME*, shall be equal; which was to be prov'd.

...y means of the latter Construction, we can easily find a very sim- ... Equation having but one unknown Quantity therein, which Equa- ... being constructed by any Conick Section at pleasure, according to ... Rules prescrib'd in Book VIII. the Value of the unknown Quan- ... will determine the Solution of the Problem. For Example: From ... Point *M* draw the Line *MP* parallel to the Asymptote *CK*, and ...eting the other Asymptote *CH* in the Point *H*; then call the given ...uantities *AM*, a; *AK*, b; *CK*, c; and the unknown ones *AP*, x; ...*M*, y. This being done, by the Nature of the Circle, we shall have ...is Equation $xx+yy=aa$, and by * the Nature of the opposite Hyper- ...la's $CH \times HM$ $(xy-cx-by+bc) = CK \times KA$ (bc); whence ... $-cx-by$ is $=0$, and so y is $=\frac{cx}{x-b}$. Now if the Square of this Va- ...e of y be put for yy in the Equation $xx+yy=aa$, by due working ...e shall have this Equation of the fourth Degree, *Art. 100.

$$x^4 - 2bx^3 + bbxx + 2aabx - aabb = 0.$$
$$\begin{array}{l} +cc \\ -aa \end{array}$$

And

And if from (C) the Centre of the Hyperbola's, there be drawn the Line CG perpendicular to AC, meeting the Circumference in the Point G; the right-angl'd Triangles ACG, AKC, will give these Equalities $\overline{CG}^2 = \overline{AG}^2 - \overline{AC}^2 = \overline{AM}^2 (aa) - \overline{AK}^2 (bb) - \overline{CK}^2 (cc)$. Whence if the given Quantity CG be called m, the preceding Equation may be chang'd into this $x^4 - 2bx^3 - mmxx + 2aabx - aabb = 0$, wherein the given Quantities are the Radius AM (a), the Lines AK, (b), CK (c), CG (m), and the unknown Quantity x expresses the Values of AP being such that drawing the Perpendiculars PM, these shall meet the Circumference in the Points sought.

In order to know which of the two Points, wherein each Perpendicular PM cuts the Circumference of the Circle, is that requir'd in the Problem; you must draw PM on the same Side the Line AP, as the Point M was suppos'd to fall in performing the Process. When $\frac{cx}{x-b}$ the Value thereof, which we have found above, is positive, that is, when x is affirmative, and greater than b; and contrariwise, PM must be drawn on the other Side AP when its Value is negative, that is, when x is affirmative, but less than b.

Another Way.

FIG. 257. Through the sought Point M, (esteem'd as given) draw the right Line MD perpendicular to the Radius AM, and through the Point D, wherein it meets AF, draw the right Line GH parallel to AM, meeting the Line MF in H, and the Line EM (produced) in G; then because the Triangles FAM, FDH are similar, therefore $AM : DH :: AF : FD$. And because the Triangles CAM, CDG, are similar, therefore $AM : DG :: AC : CD$. But the Line DG is equal to DH, because, according to the Conditions of the Problem, the Angles AME, AMF, must be equal to one another, and so the Angles DMH, DMG are equal. Therefore $AF : FD :: AC : CD$ and $AF + FD : AF :: AC + CD$ or $AD : AC$. This being premised, draw EB, MP, perpendicular to AF, and MQ to EB, and call the given Quantities AM, a; AB, b; BE, c; AF, d; and the unknown ones AP, x; PM, y. Now because the right-angl'd Triangles APM, AMD, are similar, therefore $AP (x) : AM (a) :: AM (a) : AD = \frac{aa}{x}$. And so FD is $= d - \frac{aa}{x}$. Again, because the Triangles EQM, MPC, are similar, therefore EQ or $EB - MP (c-y) : QM$ or $AP - AB (x-b) :: MP (y) : PC = \frac{xy-by}{c-y}$. Whence AC or $AP + PC = \frac{cx-by}{c-y}$; and

putting

putting the literal Expressions of this proportion $AF + PD : AF :: AD : AC$, multiplying the Means and Extremes, and then we shall have this Equation $2cdxx - aacx - 2bdxy + aaby + aady = aadc$, which divided by $2dc$, and making $b + d = f$ for brevities sake, will become this, $xx - \frac{b}{c}yx - \frac{aa}{2d}x + \frac{aaf}{2cd}y - \frac{1}{2}aa = 0$, the Locus whereof which is an Hyperbola between the Asymptotes being constructed by the 339th Article, shall cut the Circumference of the Circle in the sought Point M.

If an Equation be requir'd that has only the unknown Quantity x therein, then this Equation $xx + yy$ appertaining to the Circle must be used, in which if the Square of the Value of y found by means of the preceeding Equation be substituted for yy, we shall get an Equation of the fourth Degree, containing but one unknown Quantity x, and one of the Roots thereof shall express the Value of AP sought.

EXAMPLE VIII.

439. A Circle whose Centre is A, together with the Points E, F, being given: To find the Point M in the Circumference, being such, that if the right Lines AM, MF, ME, be drawn; the right Sine of the Angle AMF, may be to the right Sine of the Angle AME, in the given Ratio of m to n. FIG. 258.

I shall resolve this Problem three different ways.

First Way.

In the given Lines AF, AE, assume AB, AD, third Proportionals to AF, AM, and to AE, AM, and from the sought Point M (supposed to be given) draw the right Lines MB, MD, and draw the Perpendiculars MG, MH, to AF, AE, and the Parallels MP, MQ, to AE, AF. Then in BM take BK equal to DM; and from the Point K draw the right Lines KO, KL, parallel to MG, MP, and from the given Point D the right Line DC perpendicular to AF. Now the Triangles BMG, BKO, are similar, therefore $BM : BK$ or $DM :: MG : KO$. And by the Nature of the Problem $m : n :: KO : MH$, because if DM be taken for the Radius or whole Sine, the right Lines KO, MH, shall be the right Sines of the Angles MBF, MDE, or of their Complements MBA, MDA, to two right Angles, equal by Construction to the Angles AMF, AME. Therefore if the Antecedents and Consequents of the said two Proportions be orderly multiplyed into one another, we shall get $m \times BM : n \times MD :: MG \times KO : KO \times MH :: MG : MH :: MP : MQ$, because the Triangles MPG, MQH, are similar. This being laid down.

Let us call the given Quantities AD, a; AC, b; CD, c; AB, d; AM, r; and the unknown Quantities AP or MQ, x; PM or AQ, y. Then becauſe the Triangles ADC, PMG, QMH, are ſimilar, PG will be $= \frac{by}{a}$, $MG = \frac{cy}{a}$, $QH = \frac{bx}{a}$, $HM = \frac{cx}{a}$, $AG = x + \frac{by}{a}$, GB or $AB - AG = d - x - \frac{by}{a}$, DH or $AQ + QH - AD = y + \frac{bx}{a} - a$. And ſince the Triangles BGM, DHM, are right-angl'd, therefore $\overline{BM}^2$ or $\overline{BG}^2 + \overline{GM}^2$ ſhall be $= xx + \frac{2b}{a}xy + \frac{bbyy}{aa} - 2dx - \frac{2bd}{a}y + dd + \frac{ccyy}{aa} = xx + \frac{2b}{a}xy + yy - 2dx - \frac{2bd}{a}y + dd$, by ſubſtituting aa for $bb + cc$ the Value thereof, becauſe ACD is a right-angl'd Triangle; and moreover, $\overline{DM}^2 = yy + \frac{2b}{a}xy + xx - 2ay - 2bx + aa$. Now by the Nature of the Circle $\overline{AM}^2$ (rr) is $= \overline{AG}^2\left(xx + \frac{2b}{a}xy + \frac{bbyy}{aa}\right) + \overline{GM}^2\left(\frac{ccyy}{aa}\right) = xx + \frac{2b}{a}xy + yy$, by putting aa for its Value $bb + cc$. Whence if rr be ſubſtituted for $yy + \frac{2b}{a}xy + xx$ in the Values of $\overline{BM}^2$ and $\overline{DM}^2$, and if (for brevities ſake) you make $rr + dd = ff$, and $rr + aa = gg$; then will BM be $= \sqrt{ff - 2dx - \frac{2bd}{a}y}$, and $DM = \sqrt{gg - 2ay - 2bx}$. Laſtly, ſubſtituting theſe Values for BM and DM in the Proportion $m \times BM : n \times DM :: MP(y) : MQ(x)$, found above, and multiplying the Means and Extremes, and then we ſhall get this Equation $mx\sqrt{ff - 2dx - \frac{2db}{a}y} = ny\sqrt{gg - 2ay - 2bx}$, and ſquaring both Sides thereof, and getting out the unknown Quantity y by means of the Equation $xx + \frac{2b}{a}xy + yy = rr$, we ſhall come to an Equation of the ſixth Degree, having but one unknown Quantity x therein, which being conſtructed according to the Preſcriptions in Book 9. gives us the Value of $AP(x)$ being ſuch, that drawing PM parallel to AE, the Point M wherein that Line meets the Circumference of the Circle, ſhall be that ſought.

H.

If you ſuppoſe $m = n$, then it is plain that the Angles MBF, MDE, ſhall be equal; and ſo likewiſe ſhall the Angles ABM, ADM, or AMF, AME; whence it appears that the preceding Problem is only a particular Caſe of this Problem.

Second Way.

Join the two given Points E, F, by a ſtrait Line, and from the given Centre A draw the right Lines AD, AP, the one perpendicular, and the other parallel to EF; alſo through the ſought Point M (ſuppoſed to be given) draw PQ parallel to AD, and from the Point M draw the Radius AM meeting EF in O, and the Lines EM, FM, upon which let fall the Perpendiculars OG, OH, and FC, EB, from the Points O, F, E. This being done, the Triangles EOG, EFC, and FEB, FOH, are ſimilar, therefore $EO : EF :: OG : FC$. And $EF : FO :: EB : OH$, and conſequently $EO \times EF : EF \times FO$, or $EO : FO :: OG \times BE : CF \times OH$, that is, in the Ratio, compounded of OG to OH, or of m to n, (ſince MO being the Radius, the right Lines OG, OH, are the right Sines of the Angle EMO, FMO, the Complements of the Angles AME, AMF, to two right Angles) and of BE to CF, or of EM to MF, becauſe the right angl'd Triangles BME, CMF, are ſimilar. Therefore $EO : FO :: m \times EM : n \times MF$. This being laid down, FIG. 259.

Call the given Quantities AD or PQ, a; ED, b; DF, c; AM, r; and the unknown Quantities AP, x; PM, y. Now the Triangles APM, ADO, are ſimilar, therefore $MP\ (y) : AP\ (x) :: AD\ (a) : DO = \frac{ax}{y}$. And ſo EO is $= \frac{by+ax}{y}$, $FO = \frac{cy+ax}{y}$. But ſince the Triangles EMQ, FMQ, are ſimilar, therefore $\overline{EM}^2$ is $= \overline{EQ}^2\ (bb+2bx+xx) + \overline{MQ}^2\ (aa-2ay+yy) = ff+2bx-2ay$ (by writing rr for $xx+yy$, becauſe APM is a right-angl'd Triangle, and making $aa+bb+rr=ff$) and $\overline{FM}^2 = \overline{FQ}^2\ (cc-2cx+xx) + \overline{MQ}^2\ (aa-2ay+yy) = gg-2cx-2ay$, by writing rr for $xx+yy$, and making $aa+cc+rr=gg$. Then if the Analytick Values now found be put in the precedent Proportion $EO : FO :: m \times EM : n \times MF$, and afterwards the Means and Extremes be multiply'd, we ſhall get this Equation, $\overline{bmy+anx}\sqrt{gg-2cx-2ay} = \overline{mcy-max}\sqrt{ff+2bx-2ay}$, both Sides of which being ſquared, and the unknown Quantity y gotten out by means of this Equation $xx+yy=rr$, and we ſhall get an Equation of the ſixth Degree, which being conſtructed, will give us the Value of $AP\ (x)$ being ſuch, that drawing the Perpendicular PM, the ſame ſhall cut the Circumference in the ſought Point M.

M. *Descartes* has resolv'd this Problem much after the same manner in his 65th Letter, *Tom.* 3 It was propos'd to him by M. *Roberval* in a manner appearing different from that we have propos'd, but in the main it comes to the same thing.

The Third Way.

FIG. 260. With AE, AF, as two Diameters, describe two Arcs ART, AST, in which lay off any two Chords AR, AS, from the Point A, being always to each other in the given Ratio of m to n, and draw the right Lines ER, FS, intersecting one another in the Point M. I say, the Curve AM, being the Locus of all the Points (M) this found, shall cut the given Circle (whose Centre is A) in the sought Point M.

For if AM be drawn, and the same be suppos'd to be the Radius, then it is plain, that the Chord AR is the right Sine of the Angle AME, and the Chord AS the right Line of the Angle AMF.

Here it is proper to observe, 1. That this Construction is attended with this particular Property, *viz.* that it succeeds when the Point M is to be found in any Curve at pleasure, as well as in the Circumference of a Circle. 2. That when two Points of the said Locus are found as directed, as near to the given Curve as possible, we need but draw that Portion of the Locus that joins these two Points; which makes the Practice of the Construction very easy. 3. That the Locus of all the Points, as M, thus found, is one of the fourth Degree, as does easily appear by the Process in the second Way of solving the Problem, because if rr be not substituted for $xx + yy$ in the Values of EM, FM, then the Equation of the Locus will be $\overline{nby + nax}\sqrt{\overline{c-x}^2 + \overline{a-y}^2} = \overline{mcy - max}\sqrt{\overline{b+x}^2 + \overline{a-y}^2}$, wherein the unknown Quantities x and y do arise to the fourth Degree, when the same is freed from Surds. 4. That according to M. *Descartes*, it is a Fault in Geometry to use a Curve too much compounded in the Solution of a Problem; so that according to him, the two former Ways of solving this Problem are to be preferr'd before the latter, because the Problem is determin'd according to those two Ways by two Loci of the third Degree; yet, in my Opinion, the Facility and Simplicity of a Construction will in some measure recompence this Defect, as will farther appear in the following Example.

EXAMPLE IX.

FIG. 261. 440. TO divide a given Scalene Triangle ABC into four equal Parts by two right Lines DE, FG, intersecting one another at right Angles in the Point H.

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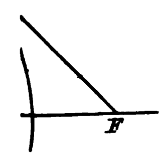
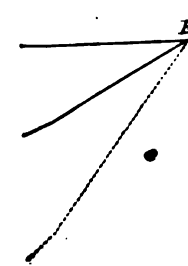




If Regard be had to the Nature of this Problem, it will appear, 1. That the two Ends D, F, of the right Lines DE, FG, must be both necessarily in the Side AC of the given Triangle, and the two other Extremities E, G, in the two other Sides BC, BA. 2. That the two sought Points D, F, must have these two Conditions; *viz.* that the Lines DE, FG, which do both of them divide the Triangle ABC into two equal Parts, do cut each other at right Angles in the Point H; and that they form the Quadrilateral Figure $BGHE$, with the two other Sides of the given Triangle, being the fourth Part of the Triangle ABC. This being suppos'd,

Draw the right Lines GI, BK, EL, perpendicular on the Side AC, and call the given Quantities AC, $2a$; BK, b; AK, c; KC, d; and the unknown Quantities AF, x; CD, y: Because the Triangle AGF, or $GI \times \frac{1}{2} AF$ is to be equal to $\frac{1}{2}$ of the Triangle ABC (ab), therefore GI is $= \frac{ab}{x}$; and by the same Reason $EL = \frac{ab}{y}$. And the similar Triangles CBK, CEL, and ABK, AGI, do give these Proportions, $BK\ (b) : EL\ \left(\frac{ab}{y}\right) :: CK\ (d) : CL = \frac{ad}{y}$. And $BK\ (b) : GI\ \left(\frac{ab}{x}\right) :: AK\ (c) : AI = \frac{ac}{x}$. And therefore DL or $CD - CL$ is $= y - \frac{ad}{y}$, FI or $AF - AI = x - \frac{ac}{x}$. But the right-angl'd Triangles DEL, FGI, are similar to one another; because each of them is similar to the Triangle FDH, which is right-angl'd at H, according to the Import of the Problem requiring the two right Lines DE, FG, to be at right Angles to each other. Whence EL $\left(\frac{ab}{y}\right) : LD\ \left(\frac{yy-ad}{y}\right) :: FI\ \left(\frac{xx-ac}{x}\right) : IG\ \left(\frac{ab}{x}\right)$; therefore multiplying the Means and Extremes, and we shall get this Equation $xxyy - acyy - adxx + aacd = aabb$, or $\overline{xx-ac} \times \overline{yy-ad} = aabb$, which takes in the first Condition of the Problem; but now to fulfil the second, *viz.* which requires the Trapezium $BGHE$ to be equal to $\frac{1}{4}$ of the given Triangle ABC.

From H the Point wherein the two right Lines DE, FG, intersect one another, draw the right Lines HA, HC, HB, to the three Angles of the given Triangle; then will $FD\ (x+y-2a) : AF\ (x) :: FHD$ $(\frac{1}{4}ab) : FHA = \frac{abx}{4x+4y-8a}$. And therefore the Triangle AHG or FGA minus the Triangle FHA is $= \frac{1}{2}ab - \frac{abx}{4x+4y-8a}$. Moreover (2.)

$AI\left(\frac{ac}{x}\right) : IK\left(\frac{cx-ac}{x}\right) :: AG : GB :: AHG\left(\frac{abx+2aby-4aab}{4x+4y-8a}\right) : GHB = \frac{bxx-5abx+2bxy-2aby+4aab}{4x+4y-8a}$. After the same Way of Reasoning, you will find that the Triangle HEB is $= \frac{byy-5aby+2bxy-2abx+4aab}{4x+4y-8a}$. Now if the Triangles HGB, HEB, be added together, the Quadrilateral Figure $HGBE$ will be formed, which must be equal to $\frac{1}{4}ab$ the fourth Part of the Triangle ABC: Whence we shall have this second Equation $xx+yy+4xy-8ax-8ay+10baa=0$.

If the unknown Quantity y be gotten out, then we shall have an Equation of the eighth Degree, which takes in all the Conditions of the Problem, and has but one unknown Quantity x therein; and so the whole Difficulty is brought to this, *viz.* to find the Roots of this Equation, which may be done by means of two Loci of the third Degree each, according to the Directions in the 417th and 418th Articles. But because the Construction of these Loci is very long and tedious, upon account of the great Number of Terms in their respective Equations, it will be much more natural to construct the Loci of the two Equations before found, though one of them be a Locus of the fourth Degree, and consequently more compound; for the other being but a Locus of the second Degree, will, together with the Facility of the Construction, make amends for this Inconveniency. Now the Construction of these Loci is thus.

FIG. 262. Draw two indefinite right Lines AB, AC, forming the right Angle BAC, and produce BA to E, so that AE be $= \sqrt{ac}$, and CA to F, so that AF be $= \sqrt{ad}$. In AC assume the Part AP of any Length, and about the Centre E, with the Distance AP, describe an Arc of a Circle, and take AH such that the Rectangle $HA \times AG$ be equal to the given Triangle BAC, and in AB assume $AQ = FH$. Again, draw the right Lines PM, QM, parallel to AB, AC, intersecting one another in the Point M; and an infinite Number of other Points, as M, being found after the same manner, draw the Curve KML through them. This being done, in AD the Diagonal of the Square $ABDC$ (having the Side AC equal to the Side (AC) of the given Triangle ABC) assume $AT = \frac{1}{4}AD$, and $DS = \frac{1}{4}AD$, and with TS, as a first Axis, being to its Parameter as 1 to 3, describe the Hyperbola OSR. Now, if from the Point M, wherein the Hyperbola is suppos'd to meet the Curve KML in the Square $ABDC$, the Perpendicular MP be drawn on AC, and if in AC, the Side of the Triangle ABC, you take $AF = AP$, and $CD = PM$; then, I say, the Points F, D, shall be such, that two right Lines FG, DE, being drawn

drawn, so as each of them do divide the Triangle ABC into two equal Parts (which is easy to do); these Lines shall cut one another at right Angles, and divide the Triangle into four equal Parts.

For if AP be called x; and PM, y; then since the Triangles EAG, FAH, are right-angl'd at A, the Square $\overline{AG}^2$ will be $= \overline{EG}^2\,(xx) - \overline{AE}^2\,(ac)$, and the Square $\overline{AH}^2 = \overline{FH}^2\,(yy) - \overline{AF}^2\,(ad)$. And because (by Construction) the Rectangle $HA \times AG$ is equal to the given Triangle BAC (ab): Therefore $\overline{HA}^2 \times \overline{AG}^2\,(\overline{yy-ad} \times \overline{xx-ac}) = aabb$. Whence the Curve KML shall be the Locus of the first Equation found above; and consequently the Property thereof shall be such, that if from any Point (M) thereof, taken within the Square $ABDC$, there be drawn the Line MP perpendicular to AC, and if in AC the Side of the given Triangle ABC, you take $AF = AP$, and $CD = PM$; the right Lines FG, DE, each of which do divide the Triangle ABC into two equal Parts, shall intersect one another at right Angles in the Point H.

Moreover, if from any Point (M) of the Hyperbola OSR, the right Line MV be drawn perpendicular to the first Axis TS, and if PM be produced, meeting the Diagonal AD in the Point X; the right-angl'd Isosceles Triangles APX, MVX, shall give these Proportions: $1 : \sqrt{2} :: AP$ or $PX\,(x) : AX = x\sqrt{2}$, and $\sqrt{2} : 1 :: MX\,(x-y) : MV$ or $VX = \frac{x-y}{\sqrt{2}}$; and therefore AV or $AX - XV = \frac{x+y}{\sqrt{2}}$. But by Construction AD is $= 2a\sqrt{2}$ because AC is $= 2a$, and consequently TS or $DT - DS = \frac{2}{3}a\sqrt{2}$. Therefore TV or $AV - AT = \frac{x+y-2a}{\sqrt{2}}$, and VS or $TV - TS = \frac{3x+3y-10a}{3\sqrt{2}}$, and by the Nature of the Hyperbola $TV \times VS\left(\frac{3xx+6xy+3yy-16ax-16ay+20aa}{6}\right) : \overline{MV}^2\left(\frac{xx-2yx+yy}{2}\right) :: 1 : 3$, that is, as the first Axis TS to the Parameter thereof: Whence by multiplying the Means and Extremes, we shall get this Equation $xx + yy + 4xy - 8ax - 8ay + 10aa = 0$, whose Locus shall be the Hyperbola OSR, being attended with this Property; *viz.* That if from any Point (M) thereof (taken within the Square $ABCD$) you draw MP perpendicular to AC, and if in AC the Side of the given Triangle ABC, you take $AF = AP$, and $CD = PM$; the right Lines FG, DE, which do each divide the Triangle ABC into two equal Parts, shall also divide the same Triangle into four equal Parts.

Now because the Point M is both in the Curve KML and in the Hyperbola OSR; therefore the Points D, F, taken in AC the Side of the given Triangle, shall have the two Conditions requir'd in the Problem.

If the Curves OSA, KML, should not meet one another within the Square $ABDC$; then we may be sure that we have made a false Supposition, *viz.* In conceiving the two Extremities D, F, to meet one another in the Side AC. Therefore they must be suppos'd to be in one of the other Sides, and the Process begun again, (by reasoning after the same manner as before) in order to have a Construction relating to this last mention'd Side. But if Regard be had to the three Remarks following, you will find it easy to take that Side of the given Triangle, wherein the two Extremities D, F, must fall, that so you may have no need of beginning the Process again.

The first Remark is, that $\overline{CL}^2$ is $= \frac{aabb}{4aa-ac} + ad$, and $\overline{BK}^2 = \frac{aabb}{4aa-ad} + ac$; which appears to be so, by substituting AC $(2a)$ for its Value AP (x) in the Equation $yy = \frac{aabb}{xx-ac} + ad$, and by putting AB $(2a)$ for its Value AQ (y) in $xx = \frac{aab}{yy-ad} + ac$. The second Remark consists in this, *viz.* that CR is $= \sqrt{2aa} = BO$; which will be found so by first substituting AC $(2a)$ for its Value AP (x) in the other Equation, $xx+yy+4xy-8ax-8ay+10aa=0$, whose Locus is the Hyperbola OSR, and afterwards AB $(2a)$ for its Value AQ (y). The third Remark is taken from hence, *viz.* if AK (c) be suppos'd less than CK (d) as it is here, then will $\overline{BK}^2$ $\left(\frac{aabb}{4aa-ad} + ac\right)$ be less than $\overline{CL}^2$ $\left(\frac{aabb}{4aa-ad} + ad\right)$. Now this being premis'd, if you require $\overline{BK}^2$ $\left(\frac{aabb}{4aa-ad} + ac\right)$ to be less than $\overline{BO}^2$ $(2aa)$, then by substituting $2a-c$ for its Value d, and duly working you will find that $bb+cc$ must be less than $4aa$, that is, the Side (AB) of the given Triangle ABC must be less than the Side AC; and if you require $\overline{CL}^2$ $\left(\frac{aabb}{4aa-ac} + ad\right)$ to be greater than $\overline{CR}^2$ $(2aa)$, then by substituting $2a-d$ for its Value c, and duly working you will find that the Side BC $(\sqrt{bb+dd})$ must be greater than the Side AC $(2a)$. But because BK is less than BO, and CL greater than CR; it is evident, that the Curves KML, OMR, will necessarily intersect each other with-

within the Square $ABDC$. Hence, if all the three Angles of the given Triangle ABC be acute, and you take for the Side AC, in which the two Points F, D, are suppos'd to meet one another, that of the three having a mean Bigness between the two others, and the shortest Side for AB, the Problem shall always necessarily have a Solution; because then the Point K (Fig. 261.) will always fall between the Points A, C; and AK is less than AC, as is supposed in performing the Calculus upon which all this reasoning is founded. After the same manner if the given Triangle be right or obtuse-angl'd, and you take the mean Side for the Side AC, in which the two Extremities D, F, must fall, the Problem will always have a Solution; so that this is a general Remark for all Sorts of Triangles.

It appears in the 262d Figure, that the Hyperbola OSR, and the Curve KML do cut one another not only in the Point M within the Square $ABDC$, as the Problem requires; but moreover, in another Point M without the Square. And by means of this latter Point we can solve the following Problem, whereof this here is but a particular Case.

To find two Points F, D, in the Side AC of a given Triangle ABC, being such, that drawing the right Lines FG, DE, forming the Triangles FGA, DEC, with the two other Sides AB, BC, each equal to $\frac{1}{4}$ of the Triangle ABC; the said Lines FG, DE, shall intersect each other at right Angles in the Point H, and the Quadrilateral Figure $BGHE$ equal to $\frac{1}{4}$ Part of the Triangle ABC. Fig. 263.

For when the Point of Intersection (M) does fall within the Square $ABDC$, it is manifest that each of the Lines AP, PM, will be less than the Side AC, and so the Points F, D, determined by them, shall both fall between the Points A, C; and will solve the Problem as proposed at first. But when the Point M does fall without the Square, because one of the Lines AP, PM, is less than the Side AC, and the other greater; therefore one of the Points F, D, does fall in the Side AC of the given Triangle, and the other in the said Side continued out; and by this means we shall get a Solution of the Problem as proposed just now. Fig. 262, 263.

EXAMPLE X.

441. A Conick Section MAM being given, together with a Point S without the Plane thereof, for the Vertex of the Cone of which it is the Section: It is requir'd to find the Position of the Circle MnN which is the Base of the Cone.

This Problem may be distinguished into two Cases, the first whereof is, when the given Section is a Parabola, and the second, when the same is an Ellipsis or Hyperbola.

 Case 1.

FIG. 264. *Case* 1. The Problem amounts to this, *viz.* to find the Point A in a Parabola, being such, that drawing the Diameter AP from that Point, together with the Line AS; the Line SD from the Point S parallel to AP; an Ordinate PM from any Point P to the Diameter AP, in the Plane of the Parabola, and a Perpendicular aD to that Ordinate in the Plane of the Triangle DSA, meeting the Sides SA, SD, in the Points a, D: The Square of PM may be equal to the Rectangle $aP \times PD$. For if a Circle be described in the Plane aPM, having aD for a Diameter, then it is plain that this Circle shall pass through the Point M; because the Angle APM is a right Angle, and $\overline{PM}^2$ is $= aP \times PD$, which is an essential Property of the Circle; therefore if the Diameter PA be drawn, and if from D the Extremity of the Diameter Da, the Line DS be drawn parallel to PA, meeting aA drawn from the other Extremity a, through A the Origin of the Diameter AP, in the Point S; the Section of the Cone, having S for its Vertex, and the Circle MAN for the Base, made by the Plane *Art. 269. APM, will be * the given Parabola MAN. Now the Point A may be determin'd thus.

Call the unknown Parameter of the Diameter AP, v; then by the Nature of the Parabola $\overline{PM}^2 = AP \times v$: But the Problem requires that $\overline{PM}^2$ be $= aP \times PD$. Whence $aP \times PD = AP \times v$: and so $AP : Pa :: PD : v$, and drawing AO parallel to Da, then will $SO : AO :: PD$ or $AO : v$, and therefore $SO \times v = \overline{AO}^2$.

Now to find the analytick Values of those Lines, from the given Point S, draw the Line SF perpendicular to the Plane of the Parabola, and from the Point F wherein it meets this Plane, draw the Line FG perpendicular to the Axis BG, and meeting the Diameter AP in H. From the Point A draw the Ordinate AK to the Axis, and the Line AQ perpendicular to the Tangent AL, which meet the Line FQ drawn through the Point F parallel to the Axis, in the Points E and Q. Lastly, from the Point Q raise QO perpendicular to the Plane of the Parabola, which will meet SD in the same Point O, as the Line AO (parallel to AD) meets it in. For the Tangent AL being parallel to the Ordinate PM which is perpendicular to aD, the Angle LAO shall be a right Angle, as well as the Angle LAQ, and so the Plane QAO shall be perpendicular to AL, and to the Plane of the Parabola passing through AL; whence the Line QO being perpendicular to this Plane will be in the Plane QAO, and consequently shall meet the Line SD in the same Point O, as the Plane QAO does, that is, in the same Point as the Line AO parallel to aD meets it. Here observe that all these Lines except FS, QO, are in the Plane of the Parabola. This being premised,

2 Call

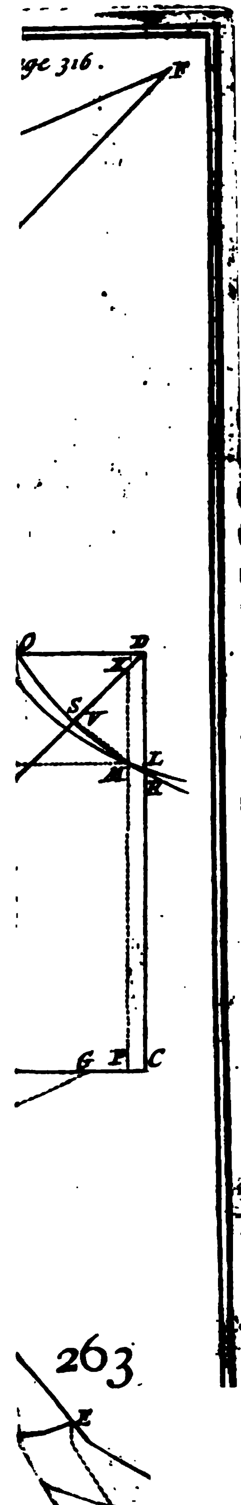
Call the given Quantities SF or QO, a; FG or KE, b; GB, c; the Parameter of the Axis, p; and the unknown Quantities BK, x; KA or GH, y. Now because the Triangles AKT, AEQ, are similar, therefore $AK\ (y) : KT\ (\frac{1}{2}p) :: AE\ (b+y) : EQ = \frac{bp}{2y} + \frac{1}{2}p$: And so since the Triangles AEQ, AQO, are right-angl'd at E and Q, $\overline{AO}^2$ or $\overline{AE}^2 + \overline{EQ}^2 + \overline{QO}^2 = \frac{bbpp}{4yy} + \frac{bpp}{2y} + \frac{1}{4}pp + bb + 2by + yy + aa$. And the Parameter of the Diameter AP, *viz.* v is $=$ * $p + 4x = p + \frac{4yy}{p}$ by substituting $\frac{yy}{p}$ for x; **Art.* 17.
and SO or FQ or $GB + BK + EQ = c + x + \frac{bp}{2y} + \frac{1}{2}p = c + \frac{yy}{p} + \frac{bp}{2y} + \frac{1}{2}p$; now if these literal Values be substitnted for the Lines they express in the Equation $\overline{AO}^2 = SO \times v$, then will $\frac{bbpp}{4yy} + \frac{bpp}{2y} + \frac{1}{4}pp + bb + 2by + yy + aa = cp + yy + \frac{bpp}{2y} + \frac{1}{2}pp + \frac{4cyy}{p} + \frac{4y^4}{pp} + 2by + 2yy$; and striking out the same Quantities from both Sides, substituting px for yy, and afterwards duly ordering the Equation, we shall get

$$\begin{array}{llll} x^3 + c\,xx & + \frac{1}{4}cp\,x & - \frac{1}{16}bbp & = 0 \\ \quad + \frac{1}{2}p & - \frac{1}{4}aa & & \\ & - \frac{1}{4}bb & & \\ & + \frac{1}{16}pp & & \end{array}$$

and the Value of x which may be found * by means of the given Parabola, will give us BK, by which we can find the Point A required. **Art.* 387

Case 2. Here the Problem may be expressed thus: To find the Point A in the given Hyperbola MAN being such, that if the Diameter AB, as also the right Lines SAa, BSb be drawn; and if thro' any Point P of the Diameter AB an Ordinate PM in the Plane of the Hyperbola, and a Perpendicular ab to this Ordinate in the Plane of the Triangle aSb: The Square $\overline{PM}^2$ may be equal to the Rectangle $aP \times Pb$. This is proved like as in the Parabola, and we must proceed thus for finding the Point A. Fig. 265

Let v be the Conjugate Diameter to AB, and in the Plane of the Triangle aSb draw the Lines AO, OZ, parallel to ab, AB, meeting


ge 316.
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L
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G F C
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$\frac{2bffxx+2cffxy}{ddxy+ffxy-bffx+cddy}$. Whence $\frac{GR \text{ or } DK+KR \times SF}{DG} = \frac{ddxy+ffxy+bffx-cddy}{ddxy+ffxy-bffx+cddy}$, and $\frac{DR \text{ or } DK+KR \times FS}{DG} = \frac{2addxy+2affxy}{ddxy+ffxy-bffx+cddy} = QO$, because $DG : DR :: BF : BQ :: FS : QO$; and since the Triangle AQO is right-angl'd, therefore $\overline{AO}^2$ or $\overline{AQ}^2 + \overline{QO}^2 = \frac{\overline{2bffx+2cffy}|^2 \times \overline{ddxx+ffxx-d^4}| + \overline{2addxy+2affxy}|^2}{\overline{ddxy+ffxy-bffx+cddy}|^2}$. Moreover, $SO : SB :: FQ : FB :: GR : GD$, and $vv = {}^{*}4xx+4yy+4ff-4dd$, or * Art. 125.
$4xx-4dd+\frac{4ffxx}{dd}$.

Now if these literal Values be put for the Lines equal to them in the Proportion $SO : SB :: \overline{AO}^2 : vv$, and the Means and Extremes be multiplied, then we shall get this Equation $\overline{2bfx+2cfy}|^2 \times \overline{ddxx+ffxx-d^4}| + \overline{2addxy+2affxy}|^2 = \overline{ddxy+ffxy+bffx-cddy}| \times \overline{ddxy+ffxy-bffx+cddy}| \times \overline{4xx+\frac{4ffxx}{dd}-4dd}|$, in which all the Terms affected with y do go out; and if $\frac{ffxx}{dd}-ff$ be put for yy, then will $bbd^4x^4 + 2bbddffx^4 + bbf^4x^4 - bbd^6xx - ccd^6xx - bbffd^4xx - ccffd^4xx + ccffd^6 + ccd^8 = \overline{ddxx+ffxx-d^4-aadd-ccdd}| \times \overline{ddx^4+2ffx^4+\frac{f^4}{dd}x^4-d^4xx-2ddffxx-f^4xx}|$. Now if you make (for Brevity's Sake) $dd+ff=mm$, $bb+cc=nn$, $aa+dd+cc=rr$, the Equation may be brought to this $bbm^4x^4 - mmnnd^4xx + ccmmd^6 = \overline{mmxx-ddrr}| \times \overline{\frac{m^4}{dd}x^4-m^4xx}|$ and lastly, making $xx=dz$, we shall have the following Equation of the third Degree,

$$z^3 \left.\begin{matrix} -d \\ -\frac{drr}{mm} \\ -\frac{dbb}{mm} \end{matrix}\right\} zz \quad \left.\begin{matrix} +\frac{ddrr}{mm} \\ +\frac{nnd^4}{m^4} \end{matrix}\right\} z - \frac{ccd^5}{m^4} = 0;$$

one of the Roots whereof; *viz.* that which is greater than d, being such that if a mean Proportional be taken between that Root and d, the half of the first Axis; this mean Proportional does express the Value of CK, by means of which the sought Point A is determin'd. Note, the Roots of this Equation may be found by means of (even) the given

page 318.

omplex than his: Neither does he give the Analysis of the Case in which the Section is an Ellipsis or Hyperbola; and he thought it enough to assure himself, that the Equation, including the Conditions of the Problem, must not exceed the fourth Degree.

LEMMA I.

42. *IF from* B *the End of the Diameter* A B [*of a Circle*] *there be drawn any Chord* B D *terminating the Arc* A D *less than the Semicircumference; and if any two contiguous Arcs* E F, F G, *be taken any where at pleasure, each equal to the Arc* A D, *and the Chords* B E, B F, B G, *be drawn. I say the middle Chord* B F, *is to the Sum or Difference of the two Chords* B E, B G, *next to it, as the Radius* C B *is to the Chord* B D: *viz. to the Sum when* B *the common Origin of the Chords* B D, B E, B F, B G, *falls in neither of the Arcs* E F, F G; *and to the Difference when the same is in one of these Arcs.* FIG. 266, 267, 268.

For about F as a Centre, with the Radius FB, describe the Arc of a Circle cutting the Chord BG produced (if necessary) to H, that so we may have an Isosceles Triangle BFH which shall be similar to the Isosceles Triangle DCB; because the Measure of the Angle FBH is half of the FG equal to the Arc AD, the half of which is also the Measure of the Angle CBD. Therefore $FB : BH :: CB : BD$, and so it only remains to prove that the Line BH is the Sum of the Chords BE, BG, in the former Case, and their Difference, in the latter. To do this,

Draw the Chords EF, FG, then we shall have two similar and equal Triangles BEF, FHG. For in the former Case the Angle FHB or FHG, is equal to $FBH = FBE$, since the Arcs FG, FE, are equal; likewise the Angle BEF is equal to the Angle FGH, because the Half of the same Arc BF is the Measure of them both; and therefore the Angle GFH is equal to the Angle EFB. But the Sides FE, FG, and FB, FH, are equal to one another. Therefore the Side GH shall be equal to the Side BE. Whence, &c. FIG. 266.

In the latter Case we prove almost after the same Manner that the Triangles FHG, FBE, are similar and equal; and so the Line BH is the Difference between the Chords BG, BE. FIG. 268, 269.

LEMMA II.

443. *IF there be a Table of Expressions wherein the Number* 2 *being the first Term in the first Horizontal Row, and* x *the second; the 3d Horizontal Row* $xx - 2$ *is the Product of the second, by* x *minus the first; the 4th* $x^3 - 3x$, *the Product of the third by* x *minus the second; the fifth,* $x^4 - 4xx + 2$, *the Product of the fourth by* x *minus the third, and so on infinitely. Moreover, if there be any Arc* A R *of a Circle, divided into any Number of* FIG. 269, 270.

equal

equal Parts at pleasure in the Points D, E, F, G, &c. *I say, if the first Term* 2 *of the Horizontal Row in the Table does express the Diameter* B A, *and the second* x, *the first Chord* B D; *the third Row* $xx - 2$ *shall express the Value of the second Chord* B E; *the fourth Row* $x^3 - 3x$ *the Value of the third Chord* B F, *and so on to the last* B R; *where it must be observed that these Chords do become negative, when they are drawn on the other Side the Point* B.

A Table for dividing the Arcs of a Circle into equal Parts.

1	2
2	$1x$
3	$1xx - 2$
4	$1x^3 - 3x$
5	$1x^4 - 4xx + 2$
6	$1x^5 - 5x^3 + 5x$
7	$1x^6 - 6x^4 + 9xx - 2$
8	$1x^7 - 7x^5 + 14x^3 - 7x$
9	$1x^8 - 8x^6 + 20x^4 - 16xx + 2$
10	$1x^9 - 9x^7 + 27x^5 - 30x^3 + 9x$
11	$1x^{10} - 10x^8 + 35x^6 - 50x^4 + 25xx - 2$
12	$1x^{11} - 11x^9 + 44x^7 - 77x^5 + 55x^3 - 11x$
13	$1x^{12} - 12x^{10} + 54x^8 - 112x^6 + 105x^4 - 36xx + 2$
14	$1x^{13} - 13x^{11} + 65x^9 - 156x^7 + 182x^5 - 91x^3 + 13x$

FIG. 269. For (1.) When the Arc AR is less than the Semi-circumference ADB; if any Chord, as BF, be multiply'd by x, and the Chord BE next before it be taken from the Product, the Remainder will be the Chord BG next after BF, because by the aforegoing Lemma CB $(1) : BD\ (x) : BF : BE + BG = xBF$, and therefore BG is $= xBF - BE$. Whence, &c.

FIG. 270. 2. When the Arc AR is greater than the Semi-circumference ADB; then it is plain that B the common Origin of all the Chords will necessarily fall in one of the equal Parts (as GH) wherein the Arc AR is divided: And it may be proved as in (Case 1.) that the third Row in the Table does express the Value of BE, the fourth the Value of BF, and so on to BG: But it remains to demonstrate that the Row next after that expressing the Chord BG, will not express the Value of $+BH$ but $-BH$; and moreover, that the Row next after this last does express the Value of $-BI$, and so on to $-BR$.

According to the Formation of the Table, the Row next after that expressing BG is $xBG - BF$. But by the Lemma $CB\ (1) : BD\ (x) :: BG : BF - BH$, and therefore $-BH$ is $= xBG - BF$; that is, $-BH$ is expressed by the parallel Row of the Table next that expressing the Value of BG. But according to the Formation

mation of the same Table, the Row next after that expressing $-BH$ is $-xBH-BG$ the Value of BI, because by the Lemma $xBH = BI - BG$: And moreover, the Row next after that expressing $-BI$ according to the Formation of the Table, is $-xBI+BH$ the Value of the negative Chord $-BL$, because according to the Lemma xBI is $= BL + BH$. And the same may be said of all the Chords from BL to BR; and this is what remained to be demonstrated.

COROLLARY I.

444. HENCE if the Arc AR be divided into five equal Parts, the sixth Row $x^5 - 5x^3 + 5x$ shall express the Value of the Chord BR, subtending the Arc BR the Difference between the Arc AB and the Semi-circumference ABD; if the Arc AR was divided into seven equal Parts, the eighth Row would express the Value of BR; and generally the Number of equal Parts must be augmented by Unity, in order to get that Row of the Table expressing BR: Supposing the Radius $CB = 1$, the first Chord $BD = x$, and that the last Chord BR is negative, when the Arc AR is greater than the Semi-circumference. Fig. 269, 270.

COROLLARY II.

445. FROM the Composition of the Table, it appears, 1. That the Number 2 is the first Term of every perpendicular or upright Row. 2. That the Coefficients of all the other Terms of the first perpendicular Row are equal to Unity. 3. That the Coefficient of any Term of whatsoever perpendicular Row, is always equal to the Coefficient of a like Term in the perpendicular Row, to the left thereof *plus* the Coefficient of that Term which is above it: For example, the Coefficient 14 of the fourth Term $14x^3$ of the third perpendicular Row, is equal to the Coefficient 5 of the fourth Term $5x^3$ of the second Perpendicular which is on the left, *plus* the Coefficient (9) of the Term $9xx$ which is above the Term $14x^3$.

SCHOLIUM.

446. IF you should continue on dividing the Circumference into Parts equal to the Arcs AD, DE, &c. beyond the Point R; then it is manifest that those Horizontal Rows of the Table following that which does express $-BR$, would still orderly express all the negative Chords that would follow BR, until you got again beyond the Point B, when the Chords would again become negative: And so on alternately positive and negative, as often as you should pass the Point B *ad infinitum*. Fig. 270.

others the whole Circumference *plus* the Arc ABR, twice the Circumference *plus* the Arc ABR, three times the Circumference *plus* the Arc ABR, &c. The Reason of which is, because the Circumference of a Circle does return into it self, therefore it may be consider'd as a Curve making an Infinity of Revolutions about it self; therefore if the Arc AR be called d, the whole Circumference c, the Arc ABR shall be $c-d$, and we shall have the two following Series:

1. $d, c+d, 2c+d, 3c+d, 4c+d, 5c+d, 6c+d, 7c+d, 8c+d$, &c.

2. $c-d, 2c-d, 3c-d, 4c-d, 5c-d, 6c-d, 7c-d, 8c-d$, &c. orderly expressing all the Portions of the Circumference terminated by the two Points A, R. This being premis'd,

If the Arc AD be any *aliquot* Part of the Arc AR, which is less than the Semi-circumference; and a Polygon $DEFGH$, &c. (beginning at the Point D) being inscribed in the Circle, having the same Number of Sides as the Number of Times the Arc AR contains the Arc AD; and if from B, the End of the Diameter AB, you draw the Chords BD, BF, BG, BH, &c. to the Angles of the Polygon: I say, these Chords do determine like *aliquot* Parts of all the Terms of the two Series, whereof the fix'd Origin is always at the Point A.

For Example: Let $AD = \frac{1}{5}d$; then it is evident, that the Arc ADE is $= \frac{c+d}{5}$, the Arc $ADEF = \frac{2c+d}{5}$, the Arc $ADEFG = \frac{3c+d}{5}$, the Arc $ADEFGH = \frac{4c+d}{5}$, which are the fifth Parts, or the like *aliquot* Parts of the five first Terms of the first Series. And if any other Term thereof be divided by 5, then it is evident, that the Quotient will contain exactly the whole Circumference some Number of times *plus* one of the precedent five Fractions. Whence since the Chord terminating an Arc, whose Origin is in A, is the same as the Chord that terminates that Arc *plus* the Circumference taken any Number of times at pleasure; therefore the Chords BD, BE, BF, BG, BH, do determine fifth Parts of all the Terms of the first Series. After the same manner we prove, that the Arcs AH, AHG, $AHGF$, $AHGFE$, $AHGFED$, are fifth Parts of the five first Terms of the second Series, and so the Chords BH, BG, BF, BE, BD, do terminate fifth Parts of all the Terms of the second Series. But it is plain, that this Demonstration may be apply'd to any other *aliquot* Part whatsoever of the Arc AR. Whence, &c. FIG. 271.

Hence, if the two preceding Series be brought to one only, *viz.* $d, c\mp d, 2c\mp d, 3c\mp d$, &c. the two Chords next on each Side the greatest (BD) bounding the Arc AD, an *aliquot* Part of the Arc AR less than the Semi-circumference, shall bound Arcs being like *aliquot* Parts FIG. 273, 274.

For Example: Let the Arc AD be $= \frac{1}{5} AR$; I say, the odd Chords BD, BF, BG, are the affirmative Roots of this Equation $x^5 - 5x^3 + 5x = a$, and the even Chords BE, BH, are the affirmative Roots of this other Equation $x^5 - 5x^3 + 5x = -a$. If the Arc AD be $= \frac{1}{7} AR$; then the Squares of the odd Chords BD, BF, BH, shall be the affirmative Roots of this Equation $x^6 - 6x^4 + 9xx - 2 = a$, and the Squares of the even Chords BE, BK, BG, shall be the Roots of this other Equation $x^6 - 6x^4 + 9xx - 2 = -a$. FIG. 271, 273. FIG. 272, 274.

For if it be propos'd to divide the whole Circumference repeated some Number of times *plus* or *minus* the Arc AR, into equal Parts, so that the first may be less than the Semi-circumference; then, by the 444th Article it is plain, that the same Table must be form'd, as was for the Division of the Arc AR; but here the Chords must necessarily change their Signs once (before the last BR be had) when the Circumference is but once repeated, because B the common Origin of them all does fall in one of the equal Parts; twice, when the Circumference is repeated twice, because the Origin B will necessarily be found in two of the equal Parts; three times, when the Circumference is repeated thrice, because the Origin B is found in three equal Parts, and so on. Therefore the Chord BR will be positive, when the Arc AR, and the Circumference repeated some even Number of times *plus* or *less* the Arc AR, is to be divided into equal Parts; and negative, when the Circumference is repeated an odd Number of times; that is, in the former Case, the parallel Row of the Table must be made equal to $+a$; and consequently the odd Chords, or their Squares, shall be the affirmative Roots of the other Equation, whereof one of the Members is $-a$. *W. W. D.*

SCHOLIUM III.

450. THE same Things being premis'd, if the Arc AD be an odd *aliquot* Part of the Arc AR; then by the bare Inspection of the Table it appears, that all the even Terms, that is, the second, fourth, sixth, &c. except the last Term a, are always wanting in the two Equations found according to the preceding *Scholium*. But it is shewn in Algebra, that if the Signs of the even Terms of an Equation be chang'd, all that is done by this, is only changing the affirmative Roots into negative ones, and the negative ones into affirmative ones. Whence the even Chords being the affirmative Roots of the Equation, having $-a$ for one Side thereof, will become the negative Roots of the other Equation, having $+a$ for one Side of it. For Example; if the Arc AD be $= \frac{1}{5} AR$; then the odd Chords BD, BF, BG, shall be the affirmative Roots of the Equation $x^5 - 5x^3 + 5x = a$, and the even Chords BE, BH, the negative Roots thereof. FIG. 271, 273.

From

B H, *&c. that is, the shortest Chord* B F — BE + BD — B H + D G *&c.* = *o.*

For draw the Diameter *B A*, and take the Arc *A R* containing the Arc *A D* the same Number of times as the Polygon has Sides; then it is evident, as we have shewn already, if the Chord *B R* be call'd *a*; and the Radius *C A* or *C B*, 1; that the odd Chords *B D*, *B F*, *B G*, &c. shall be the affirmative Roots, and the even Chords *B E*, *B H*, &c. the negative Roots of the Equation, whereof one Side is + *a*. And since the second Term, which is equal to the Sum of the Roots, as is prov'd in *Algebra*, is always wanting in that Equation; therefore, &c.

2. *If the Diameter* B A *be drawn, and the Arc* A R *be taken, containing the Arc* A D *the same Number of times as the Polygon has Sides, and the Chord* BR *be drawn: The Product* B D × B E × B F × B G × B H, *&c. of all the Chords* B D, B E, B F, B G, B H, *&c. into one another, shall always be equal to the Product of the Chord* B R *into the Radius* C A, *raised to a Power less by Unity than the Number of Chords.*

For this last Product is equal to the Member *a*; since *B R* = *a*, and the Radius *C A* is taken for Unity in the Table. And because the Term *a* is always the last Term of the Equation, whose Roots are the Chords *B D*, *B E*, *B F*, *B G*, *B H*. &c. and the last Term of an Equation, as is proved in *Algebra*, is always equal to the Product of all its Roots; therefore, *&c.*

THEOREM II.

453. *IF any Semi-circumference* A E B *be divided into an odd Number of equal Parts, whereof the two first is the Arc* A E; *the four first, the Arc* A E F, *and so on by Pairs to the last; and if the Chords* B E, BF, *&c. be drawn: I say,* FIG. 273.

1. *That* B E *the first of those Chords, minus the second* B F, *plus the third, minus the fourth, and so on to the last inclusively, is always equal to the Radius.*

2. *That the Product* [B E × B F, *&c.*] *of all the Chords into one onother, is equal to an answerable Power of the Radius. So in this Example wherein the Number of Divisions is 5, and consequently there are but two Chords* B E, B F, *we shall have* 1. B E — BF = C A. 2. BE × BF = $\overline{CA}^2$.

For in the whole Circle inscribe the regular Polygon *E F G H*, having the same Number of Sides as there are Divisions, beginging from the Point *A*; and from the other End (*B*) of the Diameter *A B* draw the Chords *BD*, *BE*, *BH*, *B F*, *B G*, &c. to all the Angles of the Polygons; and then it is manifest, 1. That *B D* the largest of these Chords is equal to the Diameter *B A*, and so the Arc *A D* being

= *o*,

$=0$, the Arc AR ſhall be ſo alſo; and conſequently the Chord BR ſhall likewiſe be equal to the Diameter BA. 2. That the Chords BE, BH; BF, BG, &c. being taken by Pairs, are equal to one another. And this being premis'd, if the aforegoing Theorem be apply'd to this particular Caſe, you will perceive this Theorem to ariſe from it. Whence, *&c.*

THEOREM III.

FIG. 272. 454. *IF any regular Polygon* DEFGHK, &c. *be inſcrib'd in a Circle, of an even Number of Sides; and if from any Point* B *in the Circumference, there be drawn the Chords* BD, BE, BF, BG, BH, BK, *&c. to all the Angles of the Polygon. I ſay,*

1. *The Sum of the Squares of the odd Chords* BD, BF, BH, *or elſe of the even Chords* BE, BG, BK, *is equal to the Square of the Radius* CB, *taken the ſame Number of times as the Polygon has Sides.*

Draw the Diameter BA, and take the Arc AR containing the Arc AD the ſame Number of times as the Polygon has Sides; then if the Chord BR be called a, and the Radius CA or CB, 1; the Squares *Art. 449. of the odd Chords BD, BF, BH, &c. ſhall * be the affirmative Roots of the Equation having one Side equal to $+a$; and the Squares of the even Chords BE, BK, BG, &c. the affirmative Roots of the other Equation having one Side equal to $-a$. But the Coefficient of the ſecond Term of both the aforeſaid Equations being equal to the Sum of their Roots, is always equal to the Square of the Radius taken the ſame Number of times as the Polygon has Sides, as appears in the Table. Therefore, *&c.*

2. *If the Diameter* BA *be drawn, and the Arc* AR *taken as many times containing the Arc* AD *as the Polygon has Sides, and the Chord* BR *be drawn; the Product* $[\overline{BD}^2 \times \overline{BF}^2 \times \overline{BH}^2$ &c.$]$ *of all the Squares of the odd Chords, is equal to the Product of* $BA \mp BR$ *into an anſwerable Power of the Radius, viz.* $BA + BR$, *when the Number of the Sides of the Polygon is barely even, and* $BA - BR$ *when it is evenly even, that is, diviſible by* 4; *and the Product* $\overline{BE}^2 \times \overline{BG}^2 \times \overline{BK}^2$, &c. *of the even Chords, is equal to the Product of* $BA \pm BR$ *into the ſame Power of the Radius, viz.* $BA - BR$ *in the former, and* $BA + BR$ *in the latter Caſe.*

For call BR, a; and the Radius CA, 1; then it is plain, that the Squares of the odd Chords BD, BF, BH, &c. are the Roots of an Equation, whoſe laſt Term will always be $2 \mp a$, that is, $BA \mp BR$; and that the Squares of the even Chords BE, BG, BK, &c. are the Roots of an Equation, whoſe laſt Term will be always $2 \pm a$, that is, $BA \pm BR$. And becauſe the laſt Term of an Equation is equal always to the Product of all its Roots, therefore, *&c.*

COROL-

COROLLARY.

455. HENCE it is evident, 1. That the Sum of the Squares of all the Chords, as well even as odd, is equal to the Square of the Radius drawn into double the Number of Sides of the Polygon, that is, $\overline{BF}^2 + \overline{BE}^2 + \overline{BD}^2 + \overline{BK}^2 + \overline{BH}^2 + \overline{BG}^2 = 12\overline{CA}^2$. 2. That the Difference between the Squares of the even Chords and the odd ones, is always equal to nothing; that is, $\overline{BF}^2 - \overline{BE}^2 + \overline{BD}^2 - \overline{BK}^2 + \overline{BH}^2 - \overline{BG}^2 = o$. 3. That the Product of the Squares of the odd Chords *plus* the Product of the Squares of the even Chords, is equal to the Quadruple of an answerable Power of the Radius, that is, $\overline{BF}^2 \times \overline{BD}^2 \times \overline{BH}^2 + \overline{BE}^2 \times \overline{BK}^2 \times \overline{BG}^2 = 4\overline{CA}^6$. 4. That the Difference between the two Products, is equal to the Double of the Chord BR multiply'd by an answerable Power of the Radius; taking notice at the same time, that the Product of the Squares of the odd Chords is greater than that of the Squares of the even Chords, when the Number of the Sides of the Polygon is barely even, and less when the Number is evenly even; that is, $\overline{BF}^2 \times \overline{BD}^2 \times \overline{BH}^2 - \overline{BE}^2 \times \overline{BK}^2 \times \overline{BG}^2 = 2BR \times \overline{CA}^5$. 5. That the Product of the Squares of all the Chords, both even and odd, shall be always equal to the Product of $\overline{BA}^2 - \overline{BR}^2 = BA \mp BR \times BA \pm BR = \overline{AR}^2$, (because ARB is a right Angle) by an answerable Power of the Radius; that is, by extracting the square Root of both Sides, the Product of all the Chords is equal to the Product of the Chord AR by a Power of the Radius less by Unity than the Number of Chords; that is, $BF \times BE \times BD \times BK \times BH \times BG = AR \times \overline{CA}^5$.

THEOREM IV.

456. *IF the Semi-circumference* ADB *be divided into any even Number of equal Parts, the first whereof let be the Arc* AD; *the first three, the Arc* ADE; *the first five, the* ADEF, *and so on by two's to the last; and if the Chords* BD, BE, BF, &c. *be drawn; I say,* FIG. 275.

1. *The Sum of the Squares of these Chords is equal to the Square of the Radius taken as many times as there are Divisions in the Semi-circumference; that is, the Number of Divisions being here 6,* $\overline{BD}^2 + \overline{BE}^2 + \overline{BF}^2$ *will be* $= 6\overline{CA}^2$.

2. *That the Product of the Squares of the Chords by one another, is equal to twice an answerable Power of the Radius. So* $\overline{BD}^2 \times \overline{BE}^2 \times \overline{BF}^2 = 2CA^6$. *And consequently* $BD \times BE \times BF = CA^3 \times \sqrt{2}$.

Uu For

Make a Table, wherein the firſt horizontal Row being 1, and the ſecond $z - 1$; let the third $zz - z - 1$ be equal to the Product of the ſecond by z leſs the firſt; the fourth $z^3 - zz - 2z + 1$, equal to the Product of the third by z, leſs the ſecond, and ſo on. Then form an Equation, one Side of which being nothing, let the other be that horizontal Row of Quantities in the Table, whoſe Exponent is half the Number of Sides of the Polygon *plus* 1: I ſay, the greateſt Root z of this Equation ſhall terminate an Arc, whoſe Chord ſhall be the Side ſought of the Polygon.

A Table for the Inſcription of regular Polygons in a Circle.

1	1
2	$z - 1$
3	$zz - z - 1$
4	$z^3 - zz - 2z + 1$
5	$z^4 - z^3 - 3zz + 2z + 1$
6	$z^5 - z^4 - 4z^3 + 3zz + 3z - 1$
7	$z^6 - z^5 - 5z^4 + 4z^3 + 6zz - 3z - 1$
8	$z^7 - z^6 - 6z^5 + 5z^4 + 10z^3 - 6zz - 4z + 1$
9	$z^8 - z^7 - 7z^6 + 6z^5 + 15z^4 - 10z^3 - 10zz + 4z + 1$
10	$z^9 - z^8 - 8z^7 + 7z^6 + 21z^5 - 15z^4 - 20z^3 + 10zz + 5z - 1$

For Example, It is requir'd to inſcribe an Heptagon in a Circle. Take the 4th Horizontal Row of Quantities in the Table, becauſe four is greater than half ſeven by 1, and making it equal to nothing, we have $z^3 - zz - 2z + 1 = 0$, and the greateſt Root z of this Equation ſhall expreſs the Value of the Chord BE, terminating the Arc AE the ſeventh Part of the whole Circumference, which may be proved thus.

Let AR be an Arc of a Circle leſs than the Semi-circumference divided into any odd Number of equal Parts in the Points D, E, F, G, &c. and from B the Extremity of the Diameter BA, draw the Chords BD, BE, BF, BG, &c. Moreover, take the Arc AS equal to the Arc AD, draw the Chord BS, and call the firſt Chord BD, or BS, x; and the ſecond BE, z. Then by the Lemma, $CB\,(1) : BE\,(z) :: BD\,(x) : BF + BS$. And conſequently $BF = xz - x$. Moreover, $CB\,(1) : BE\,(z) :: BF : BD + BH$. And conſequently $BH = zBF - BD$: In like manner $CB\,(1) : BE\,(z) :: BH : BF + BR$, and therefore $BR = zBH - BF$: That is, the fifth Chord BH is equal to the Product of the third BF by z *minus* the firſt BD; the ſeventh BR is equal to the Product of the fifth BH by z *minus* the third BF, and ſo on. Whence it appears, that if a Table be made wherein the firſt Horizontal Row [of Quantities] is x, and the ſecond $zx - x$; the third, is $xzz - xz - x$ equal to the Product of the ſecond by z *minus* the firſt; the fourth $xz^3 - xzz - 2xz + 1$ equal to the Product of the third by z *minus* the ſecond, and ſo on: Then ſhall the Rows of Quantities of FIG. 275.

 this

LEMMA I.

459. *IF* A D, E F, *be two equal Arcs of the Semi-circle* A E B, *one of them, as* A D *being taken from* A, *the end of the Diameter* A B, *and the other any where at pleasure; and if the Chords* B D, B E, B F, *and* A D, A E, A F, *be drawn: I say,* 1. $AB \times BF = BD \times BE - AD \times AE$. 2. $AB \times AF = BD \times AE + AD \times BE$. FIG. 277.

For the three right-angl'd Triangles ADG (the Point G being here the Intersection of the Chords BD, AF) AEB, BFG, are similar to each other, because the Angle AGD or BGF, having half of the two Arcs BF, AD, for the Measure thereof, is equal to the Angle BAE, having likewise the half of the two Arcs BF, FE, or AD, for the Measure thereof. Whence if the Diameter AB be called 1; the Chords BD, x; AD, y; BE, v; AE, z; then will $BE\,(v) : EA\,(z) :: AD\,(y) : DG = \frac{yz}{v}$; and therefore BG or $BD - DG = x - \frac{yz}{v}$. 2. And $AB\,(1) : BE\,(v) :: BG\left(x - \frac{yz}{v}\right) : BF = vx - yz$, that is (because $AB = 1$) $AB \times BF = BD \times BE - AD \times AE$. Which was in the first place to be demonstrated.

Now $BE\,(v) : BA\,(1) :: AD\,(y) : AG = \frac{y}{v}$. And $AB\,(1) : AE\,(z) :: BG\left(x - \frac{yz}{v}\right) : GF = xz - \frac{zzy}{v}$; and therefore $AG + GF$ or $AF = xz - \frac{zzy}{v} + \frac{y}{v} = xz + vy$, because the Triangle AEB being right-angl'd, $1 - zz = vv$; that is, $AB \times AF = BD \times AE + AD \times BE$. Which was what remained to be demonstrated.

LEMMA II.

460. *LET there be form'd a Table, the first Horizontal Row whereof consisting of two Parts* x *and* y; *let all the others be so likewise according to this Law, viz. That the first Part of any Horizontal Row be form'd by multiplying the first Part of that Horizontal Row immediately before it by* x minus *the second Part of the same multiply'd by* y; *and the second Part form'd by multiplying the said first one by* y, plus *that second Part multiply'd by* x. *Moreover, let there be any Arc* (A R) *of a Circle,* FIG. 278.
less than half the Circumference, which suppose to be divided into any Number of equal Parts at pleasure, in the Points D, E, F, G, &c. *I say, if the Diameter* AB = 1, *and the two first Chords* BD = x, AD = y; *then all the other Chords* BE, BF, BG, &c. *shall be express'd by the first Parts of the second,*

third, fourth, &c. Horizontal Row of Terms, and the other correspondent Chords A E, A F, A G, *&c. by the second Parts of the same Rows.* So B G *being the fourth Chord, will be expressed by* $x^4-6yyxx+y^4$ *the first Part of the fourth Horizontal Row, and its Correspondent* A G *by the second Part* $4yx^3-4y^3x$ *of the same Row.*

1	x	y
2	$xx-yy$	$2yx$
3	x^3-3yyx	$3yxx-y^3$
4	$x^4-6yyxx+y^4$	$4yx^3-4y^3x$
5	$x^5-10yyx^3+5y^4x$	$5yx^4-10y^3xx+y^5$
6	$x^6-15yyx^4+15y^4xx-y^6$	$6yx^5-20y^3x^3+6y^5x$
7	$x^7-21yyx^5+35y^4x^3-7y^6x$	$7yx^6-35y^3x^4+21y^5xx-x^7$

For by the foregoing Lemma it is plain that the Product of any Chord *B F* by the first Chord *B D* (x), *minus* the Product of the correspondent Chord *A F*, by the other first Chord *A D* (y), does express the Value of the Chord *B G*, being that next after *B F*; and so the Chord *A G* is $= BF \times AD\,(y) + AF \times BD\,(x)$. Whence, &c.

Corollary.

461. IF the two Parts of each horizontal Row of Terms in the aforegoing Table be added together, and all the Terms be order'd according to the different Powers of x; the following new Table will thereby be form'd, orderly containing all the Powers of the Binomial $x+y$; where the first and second Terms must be affirmative, the third and fourth negative, and so of every two alternately to the last. Thus the third horizontal Row of Terms is $x^3+3yxx-3yyx-y^3$; that is, the Cube of the Binomial $x+y$, the two first Terms whereof are affirmative, and the two others negative. In like manner, the 5th horizontal Row will be $x^5+5yx^4-10yyx^3-10y^3xx+5y^4x+y^5$, which is the fifth Power of the Binomial $x+y$, whereof the first and second Terms are affirmative, the third and fourth negative, and the fifth and sixth affirmative.

1	$x+y$
2	$xx+2xy-yy$
3	$x^3+3yxx-3yyx-y^3$
4	$x^4+4yx^3-6yyxx-4y^3x+y^4$
5	$x^5+5yx^4-10yyx^3-10y^3xx+5y^4x+y^5$
6	$x^6+6yx^5-15yyx^4-20y^3x^3+15y^4xx+6y^5x-y^6$
7	$x^7+7yx^6-21yyx^5-35y^3x^4+35y^4x^3+21y^5xx-7y^6x-y^7$

For

For if Regard be had to the manner of the Formation of the preceding Table, it will appear, that all the Terms of every horizontal Row are form'd by those of that horizontal Row next before it, multiply'd by x and y, and so connected together by the Signs + and —, so that the Terms of the two Parts composing every horizontal Row, being put in order, according to the different Powers of the unknown Quantity x, there is an alternate Succession of two affirmative, and then two negative Signs.

SCHOLIUM.

462. IN this latter Table it is manifest, that the Coefficients of every Term of the first perpendicular or upright Row, is Unity; of the Coefficients of the second Row, the natural Numbers 1, 2, 3, 4, &c. being form'd by the continual Addition of Unities; the Coefficients of the third Row, the triangular Numbers, 1, 3, 6, 10, &c. being form'd by the continual Addition of the natural Numbers; the Coefficients of the fourth Row, the Piramidal Numbers 1, 4, 10, 20, &c. being form'd by the continual Addition of triangular Numbers; and so on from Row to Row towards the right Hand, the Numbers of a superior Order being formed by the continual Addition of those of the Order immediately going before.

EXAMPLE XII.

463. AN Arc of a Circle AR being given; to divide it into any Number of equal Parts at pleasure in the Points D, E, F, G, &c. after a different manner from that in the 10th Example. FIG. 278.

Call the Diameter AB, 1; the given Chords BR, AR (terminating the given Arc AR) a and b; and the unknown Chords BD, AD, (terminating the sought Arc AD) x and y, and raise the Binomial $x+y$ to such a Power, that its Exponent be equal to the Number of Divisions. Then form two Equalities; one between the given Quantity a, and all the odd Terms of that Power of $x+y$ connected by the Signs + and — alternately; and the other, between the given Quantity b, and all the remaining Terms of the said Power of the Binomial $x+y$, connected by the alternate Signs. This being done, get out one of the unknown Quantities x or y by means of the Equations $xx = 1 - yy$, or $yy = 1 - xx$, arising from the right-angl'd Triangle ADB; and then we shall have an Equation, having only one unknown Quantity in it, which being resolv'd, will give us BD or AD, terminating the Arc sought AD.

For Example: It is requir'd to divide the given Arc AR into seven equal Parts in the Points D, E, F, G, H, I. Take the seventh Power of the Binomial $x+y$, *viz.* $x^7 + 7yx^6 + 21yyx^5 + 35y^3x^4 + 21y^5xx + 7y^6x$

+

For supposing a, c, d, e, to express the four first Cells of the second upright Row, and a, f, g, b, the four first Cells of the third Row; then by the Formation of the Cells, we shall have $b = e + g$, $g = d + f$, $f = c + a$, and therefore $b = e + d + c + a$; which is the thing to be prov'd. And it is manifest, that this Demonstration extends to any Number of Cells whatsoever of two adjoining upright Rows. Whence, &c.

COROLLARY.

465. BEcause all the Cells, the first horizontal and upright Rows being excepted, are composed of two Terms, the Letter a being in the first, and b in the last Term; therefore, 1. The Term wherein is the Letter a, is equal to the Term wherein is likewise found the Letter a in the next Cell to the left, *plus* all the Terms wherein it is, in the Cells being above this here. 2. The Term wherein is the Letter b, is equal to the Term wherein is likewise the Letter b, in the Cell to the left, *plus* all the Terms wherein it is found in the Cells above it. So the Term $15\,a$ of the fifth Cell of the fourth upright Row, is equal to the Term $5\,a$ of the Cell to the left, *plus* the Terms $4\,a$, $3\,a$, $2\,a$, $1\,a$, which are in the Cells above it; and $20\,b$ is equal to the Term $10\,b$ of the Cell to the left, *plus* the Terms $6\,b$, $3\,b$, $1\,b$, of all the Cells above it.

LEMMA II.

466. *IF the Term wherein is the Letter* a, *in any Cell, be multiplied by the Sum of the Index's of its horizontal and upright Rows* minus 2, *and the Product be divided by the Index of its perpendicular Row* minus 1; *I say, the Quotient shall be equal to that Term,* plus *all them that be above it. For Example: If the Term* 15 a *in the fifth Cell of the fourth upright Row, be multiply'd by* $5 + 4 - 2 = 7$, *and the Product divided by* $4 - 1 = 3$, *the Quotient* 35 a *shall be equal to the Term* 15 a, plus *the Terms* 10 a, 6 a, 3 a, 1 a, *being above it.*

This appears plain in all the Cells of the second upright Row, since they all do contain the same Term $1\,a$. And I shall demonstrate, that supposing this Property to happen in any upright Row whatsoever, it must needs be so likewise in the Cell to the right of that; from whence it will follow, that because it happens in the second upright Row, it shall be also in the third; and because in the third, it shall be in the fourth; and so on *ad infinitum*. And to prove this,

Let a, b, c, d, e, f, &c. be any Number of the Terms wherein is the Letter a, in any upright Row; and a, g, h, k, l, &c. the like Number of Terms of the Row to the right thereof. Also let m be equal to the Sum of the Indexes (*minus* 2.) of the upright and horizontal Rows

wherein is the Term f; and r equal to the Index minus 1 of the upright Row, in which that Cell is. Now by Supposition $\frac{m}{r}f = f + e + d + c + a = l$, $\frac{m-1}{r}e = e + d + c + a = k$, $\frac{m-2}{r}d = d + c + a = b$, $\frac{m-3}{r}c = c + a = g$, $\frac{m-4}{r}a = a$. Whence $l+k+b+g+a = \frac{m}{r}f + \frac{m-1}{r}e + \frac{m-2}{r}d + \frac{m-3}{r}c + \frac{m-4}{r}a = \frac{m}{r} \times f + e + d + c + a, - \frac{1}{r}\overline{1e+2d+3c+4a} = \frac{m}{r}l - \frac{1}{r} \times k + b + g + a$, by substituting l for $f + e + d + c + a$, and $k + b + g + a$, for $1e + 2d + 3c + 4a$; then if l and $-\frac{1}{r} \times k + b + g + a$ be transpos'd, we shall have $\frac{r+1}{r} \times k + b + g + a = \frac{m-r}{r}l$; and multiplying both Sides by r, dividing by $r + 1$, and adding l to both Sides, there arises $\frac{m+1}{r+1}l = l + k + b + g + a$. But because the Index of that upright Row, wherein is the Cell, whose Term is represented by l, does exceed the Index of that wherein is the Cell, whose Term is express'd by f, and these Cells are both in the same horizontal Row; therefore the Property to be prov'd in every Term, having the Letter a in any upright Row, does also agree to the Term a in the upright Row to the right thereof. Farther, because this Demonstration is the same, be the Number of Terms of two adjoining Rows what they will, therefore what we have demonstrated with regard to the Term l, will be likewise true of any Term in the same Row.

Now if we suppose n to express generally the Index of any horizontal Row except the first, then the first Cell of this Row will not contain any Term wherein is the Letter a; the second will always contain $1a$; and if you multiply $1a$ by $\frac{n+2-2}{2-1} = \frac{n}{1}$, we shall get $\frac{n}{1}a$ for the Term wherein is found the Letter a in the third Cell; and if you multiply $\frac{n}{1}a$ by $\frac{n+3-2}{3-1} = \frac{n+1}{2}$, we shall have $\frac{n}{1}a \times \frac{n+1}{2}$ for the Term wherein is the Letter a in the fourth Cell; so that this Series 0, $1a$, $\frac{n}{1}a$, $\frac{n}{1} \times \frac{n+1}{2}a$, $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+1}{3}a$, $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times$

$n+3$

$\frac{n+2}{4}a$, &c. will orderly expreſs all the Terms wherein is the Letter a in the horizontal Row of Cells, whoſe Index is n. So if $n = 5$, the Series o, $1a$, $5a$, $15a$, $35a$, &c. will orderly expreſs all the Terms, wherein is the Letter a in the 5th horizontal Row of Cells.

LEMMA III.

467. *If the Term, wherein is the Letter* b *in any Cell, be multiply'd by the Sum of the Indexes of its horizontal Row, and its perpendicular Row* minus 2, *and if the Product be divided by the Index of its perpendicular Row: I ſay, the Quotient ſhall be equal to that Term* plus, *all thoſe being above it. For Example; If the Term* 10 b *of the fifth Cell in the third upright Row be multipy'd by* $5+3-2=6$, *and the Product be divided by* 3, *then will* 20 b *be the Sum of the Term* 10 b, *together with all the Terms* 6 b, 3 b, 1 b, *above it.*

It is manifeſt, that this Property happens in the firſt perpendicular Row, wherein all the Cells do contain $1b$, except the firſt, which has not the Letter b in it. By means of which we can demonſtrate, as in the laſt *Lemma* with regard to the Terms multiplying a, that the ſame will likewiſe happen in the ſecond upright Row, the third, the fourth, and ſo in all the others *ad infinitum.* From whence we conclude, that if n expreſſes the Index of any horizontal Row, except the firſt, the Series $1b, \frac{n-1}{1}b, \frac{n-1}{1}\times\frac{n}{2}b, \frac{n-1}{1}\times\frac{n}{2}\times\frac{n+1}{3}b, \frac{n-1}{1}\times\frac{n}{2}\times\frac{n+1}{3}\times\frac{n+2}{4}b$, &c. ſhall orderly expreſs all the Terms, wherein is the Letter b in the parallel Row of Cells whoſe Index is n. So if $n = b$, the Series $1b$, $4b$, $10b$, $20b$, $35b$, &c. ſhall orderly expreſs all the Terms wherein b is, in the fifth parallel Row.

COROLLARY.

468. It follows from the two laſt Lemmata, if all the Terms of this Series be added to the Terms of that in the laſt Lemma, that thereby we ſhall form this, $1b$, $1a$, $+\frac{n-1}{1}b, \frac{n}{1}a + \frac{n-1}{1}\times\frac{n}{2}b, \frac{n}{1}\times\frac{n+1}{2}a + \frac{n-1}{1}\times\frac{n}{2}\times\frac{n+1}{3}b, \frac{n}{1}\times\frac{n-1}{2}\times\frac{n+2}{3}a\times\frac{n-1}{1}\times\frac{n}{2}\times\frac{n+1}{3}\times\frac{n+2}{4}b$, &c. or by abbreviating the Expreſſion, $b, a+\frac{n-1}{1}b, a+\frac{n-1}{2}b\times\frac{n}{1}a+\frac{n-1}{3}b\times\frac{n}{1}\times\frac{n+1}{2}, a+\frac{n-1}{4}b\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}$ &c. which ſhall orderly expreſs all the Cells of that horizontal Row of the Table whoſe Index is n.

Whence

this 1, $\frac{n+1}{1}$, $\frac{n+3}{2}\times\frac{n}{1}$, $\frac{n+5}{3}\times\frac{n}{1}\times\frac{n+1}{2}$, $\frac{n+7}{4}\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}$, &c. by means whereof at once may be found the Coefficient of the Table in *Art.* 443. its upright Row, and the Exponent of the horizontal Row being given. The Rule is this.

Take the Term in this Series answering to the given upright Row, that is, the third, if it be the third Row; the fourth, if it be the fourth, *&c.* and having substituted the Number expressing the parallel Row the Term is in, for n in that Term, you will have the Coefficient sought. For Example: If you want the Coefficient of the fourth Term $14x^3$ of the third upright Row, substitute the Number 4 for n in the third Term $\frac{n+3}{2}\times\frac{n}{1}$, and 14 will be the Coefficient sought.

For, the Exponent of the upright Row of Coefficients in the Table of *Article* 443, is the same as the Exponent of the upright Row in the Square of Cells of the last *Article*; and the Exponent of the horizontal Row, is equal to the Exponent of the horizontal Row of the Square of Cells. Whence it is manifest, that this Rule is only an Application of that of *Article* 468, to this particular Case wherein $a=2$ and $b=1$.

LEMMA V.

471. *If* 1 *be substituted for* b, *in the Square of Cells of Article* 464; *then will that be chang'd into this, wherein I say that the upright Rows do orderly contain all the Numbers call'd Figurate,* viz. *the first Row, the Numbers of the first Order, which are Units; the second Row, the natural Numbers, or those of the second Order; the third Row, the triangular Numbers, or those of the third Order, being form'd by the continual Addition of the natural Numbers; the fourth, the Pyramidal Numbers, or those of the fourth Order, being form'd by the continual Addition of the triangular Numbers; and so on to Infinity.*

	1.	2.	3.	4.	5.	6.	7.
1.	1	1	1	1	1	1	1
2.	1	2	3	4	5	6	7
3.	1	3	6	10	15	21	28
4.	1	4	10	20	35	56	84
5.	1	5	15	35	70	126	210
6.	1	6	21	56	126	252	462
7.	1	7	28	84	210	462	924

For by the 464th Article, every Cell is equal to that to the left thereof *plus*, all the others being above it.

Mr. *Paschal*

Mr. Pascal has written a Treatise, entitled, Triangular Arithmetick, wherein the Properties of these Numbers are handled, and their great Use in many Arithmetical Questions is shewn.

Corollary.

470. If you make $a = 1$ and $b = 1$ in the general Series of Article 468. a, $a + \frac{n-1}{1} b$, $a + \frac{n-1}{2} b \times \frac{n}{1}$, $a + \frac{n-1}{3} b \times \frac{n}{1} \times \frac{n+1}{2}$, $a + \frac{n-1}{4} b \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$, &c. the same will be changed to this, 1, n, $\frac{n}{1} \times \frac{n+1}{2}$, $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$, $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$, by means of which we can at once find any figurate Number requir'd, its Order being given, and the Number expressing the Place of it. The Rule is thus:

Take that Term in this last Series answering to the given Order, viz. the third, if it be the third Order; the fourth, if it be the fourth Order, &c. and having substituted the Number expressing the Place of the figurate Number for n, that is, 4 if it be 4, 5 if it be 5, and so on; and the said Number will be had. For Example: To find the fifth Number of the fourth Order, substitute 5 for n in the fourth Term $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$ of the Series, and 35 will be the Number sought.

This is only the Application of the Rule in Article 468. to this particular Case.

Problem I.

471. *It is required to find a general Series orderly expressing all the Terms in the Table for finding of Arcs in the 443d Article.*

Because the third Term of any upright Row of that Table, always answers to the first Term of the Row to the right; therefore if $m + 1$ generally expresses the Exponent of an horizontal Row, you must find the Coefficient of the Term, whose Place is express'd by $m + 1$ in the first upright Row; the Coefficient of the Term, whose Place is expressed by $m + 1 - 2$, or $m - 1$, in the second upright Row; the Coefficient of the Term, whose Place is express'd by $m - 1 - 2$, or $m - 3$, in the third upright Row, and so on. The Expression of the Place continually lessening by 2, as the upright Row goes on to the right. Whence, according to the 470th Article, you must substitute the Number $m - 1$ for n, in the second Term $\frac{n+1}{1}$; the Number

$m - 3$

$m-3$ for n in the third Term $\frac{n+3}{2}\times\frac{n}{1}$; the Number $m-5$ for n in the fourth Term $\frac{n+5}{3}\times\frac{n+1}{2}$; the Number $m-7$ for n in the fifth Term $\frac{n+7}{4}\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}$, *&c.* and so we shall have the following Series of Coefficients, 1, m, $\frac{m}{2}\times\frac{m-3}{1}$, $\frac{m}{3}\times\frac{m-5}{1}\times\frac{m-4}{2}$, $\frac{m}{4}\times\frac{m-7}{1}\times\frac{m-6}{2}\times\frac{m-5}{3}$, *&c.* and because the Signs of any horizontal Row of Terms in the Table are always alternate; and the first Term is always the unknown Quantity x raised to such a Power, whose Exponent is less by Unity than that of the horizontal Row; and since all the other Terms do contain Powers of x, whose Exponents continually lessen by 2, supposing $x^0=1$. Therefore $x^m-mx^{m-2}+\frac{m}{2}\times\frac{m-3}{1}x^{m-4}-\frac{m}{3}\times\frac{m-5}{1}\times\frac{m-4}{2}x^{m-6}+\frac{m}{4}\times\frac{m-7}{1}\times\frac{m-6}{2}\times\frac{m-5}{3}x^{m-8}$, *&c.* shall be a general Expression for that horizontal Row in the Table, whose Index is $m+1$.

When the first Terms of these Series are had, you will easily perceive the Law that they go on with, and so may continue them as far as you please. For Example, suppose in this Series, that r expresses the Place of the Term, whose Coefficient is required, then shall the Coefficient be express'd by the general Fraction $\frac{m\times m-r\times m-r-1\times m-r-2, \&c.}{r-1\times r-2\times r-3\times r-4, \&c.}$ where the Numerator and Denominator must each have as many Terms, as the Number $r-1$ contains Units. So if $r=5$, the Coefficient of the fifth Term will be $\frac{m\times m-5\times m-6\times m-7}{4\times3\times2\times1}$: and if $r=4$, then we shall have $\frac{m\times m-4\times m-5}{3\times2\times1}$.

Here it must be observed that the Number of Terms of the aforesaid Series is always determinate, it being equal to the half (*plus* 1) of the Exponent of the Horizontal Row it expresses when that Exponent is odd, and to the half, when the same is even. For Example, it will have but three Terms, when it expresses the fifth or sixth horizontal; four, when it expresses the seventh or eight, *&c.*

PROBLEM II.

474. *TO find a general Series, orderly expressing all the Terms of any horizontal Row, in the Table of Article* 457, *for inscribing of regular Poligons.*

Because

Because the second Term of every upright Row answers to the first
* Art. 459. of that being to the right; therefore if * $m+1$ be the Exponent of any horizontal Row in that Table, the Coefficients of the four first Terms of that Row shall be 1, 1, $m-1$, $m-2$, the Coefficient of the fifth Term shall be the triangular Number whose place is expressed
* Art. 4[illegible] by $m-3$, viz. * $\frac{m-3}{1} \times \frac{m-2}{2}$; that of the sixth Row the triangular Number, whose Place is express'd by $m-4$, viz. $\frac{m-4}{1} \times \frac{m-3}{2}$; that of the seventh Term shall be the piramidal Number, whose Place is express'd by $m-5$, viz. $\frac{m-5}{1} \times \frac{m-4}{2} \times \frac{m-3}{3}$; that of the eighth Term the piramidal Number, whose Place is express'd by $m-6$, viz. $\frac{m-6}{1} \times \frac{m-5}{2} \times \frac{m-4}{3}$; that of the ninth Term shall be a Number of the 5th Order, whose Place is express'd by $m-7$, viz. $\frac{m-7}{1} \times \frac{m-6}{2} \times \frac{m-5}{3} \times \frac{m-4}{4}$; and so on, *ad infinitum*. Now if the proper Powers of z be join'd to these Coefficients, and you prefix the Sign $-$ to the second and third Terms, the Sign $+$ to the fourth and fifth; and so alternately, then we shall get the following general Series, $z^m - z^{m-1} - \overline{m-1}\, z^{m-2} + \overline{m-2}\, z^{m-3} + \frac{m-3}{1} \times \frac{m-2}{2} z^{m-4} - \frac{m-4}{1} \times \frac{m-3}{2} z^{m-5} - \frac{m-5}{1} \times \frac{m-4}{2} \times \frac{m-3}{3} z^{m-6} + \frac{m-6}{1} \times \frac{m-5}{2} \times \frac{m-4}{3} z^{m-7} + \frac{m-7}{1} \times \frac{m-6}{2} \times \frac{m-5}{3} \times \frac{m-4}{4} z^{m-8}$, &c. which expresses all the Terms in that horizontal Row of the Table, in Art. 457. whose Exponent is $m+1$: Where you must observe to take the same Number of Terms, as there are Units contain'd in $m+1$.

Problem III.

461. *To find a general Series orderly expressing the Coefficients of all the Terms in any horizontal Row of the Table of Article* 460. (*or which is the same*) *of any Power of the Binomial* $x+y$.

Let m in general be the Exponent of any horizontal Row of that Table; then it is manifest, that the Coefficients of the two first
* Art. 460. Terms of that Row shall be * 1, m, and because the second Term of every upright Row (beginning at the second) answers to the first Term of

of the Row being to the right, therefore the Coefficient of the third Term of the horizontal Row ſhall * be the triangular Number, whoſe *Art. 462. Place is expreſs'd by $m-1$, *viz.* * $\frac{m-1}{1}\times\frac{m}{2}$; that of the fourth Term *Art. 472 ſhall be the piramidal Number, whoſe Place is expreſs'd by $m-2$, *viz.* $\frac{m-2}{1}\times\frac{m-1}{2}\times\frac{m}{2}$; that of the fifth Term ſhall be a Number of the fifth Order, whoſe Place is expreſs'd by $m-3$, *viz.* $\frac{m-3}{1}\times\frac{m-2}{2}\times\frac{m-1}{3}\times\frac{m}{4}$; and ſo on *ad infinitum.* Whence the general Series requir'd will be 1, m, $\frac{m}{1}\times\frac{m-1}{2}$, $\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}$, $\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}$, &c.

COROLLARY.

476. HENCE $\overline{x\mp y}^m = x^m \mp \frac{m}{1} y x^{m-1} + \frac{m\times\overline{m-1}}{1\times2} yy x^{m-2} \ldots \mp \frac{m\times\overline{m-1}\times\overline{m-2}}{1\times2\times3} y^3 x^{m-3} + \frac{m\times\overline{m-1}\times\overline{m-2}\times\overline{m-3}}{1\times2\times3\times4} y^4 x^{m-4} \mp \frac{m\times\overline{m-1}\times\overline{m-2}\times\overline{m-3}\times\overline{m-4}}{1\times2\times3\times4\times5} y^5 x^{m-5}$, &c.

PROBLEM IV.

477. *TO find a general Equation for dividing a given Arc* (AR) *of a Circle into any Number of equal Parts.*

Let m generally expreſs the Number of equal Parts, and AD the FIG. 279. firſt of the equal Parts. Draw the Diameter AB, and the Chords BD, BR; and make CA or $CB = 1$, the given Chord $BR = a$, and the ſought Chord $BD = x$. Then will * $\mp a = x^m - m x^{m-2} + \frac{m\times\overline{m-3}}{2\times1} x^{m-4} - \frac{m\times\overline{m-5}\times\overline{m-4}}{3\times2\times1} x^{m-6} + \frac{m\times\overline{m-7}\times\overline{m-6}\times\overline{m-5}}{4\times1\times2\times3} x^{m-8}$, &c. *Art. 444, and 473. (Where a is affirmative, when the given Arc AR is leſs than half the Circumference, and negative when it is greater) be the general Equation requir'd; wherein you muſt take the Number of Terms equal to the Units contain'd in half the Number m when it is even, or equal to thoſe in its half *plus* 1, when odd; becauſe what follows will be equal to o.

As ſuppoſe m be $= 5$; then will a be $= x^5 - 5x^3 + 5x$; and if $m = 7$, then $\mp a = x^7 - 7x^5 + 14x^3 - 7x$.

Another Way.

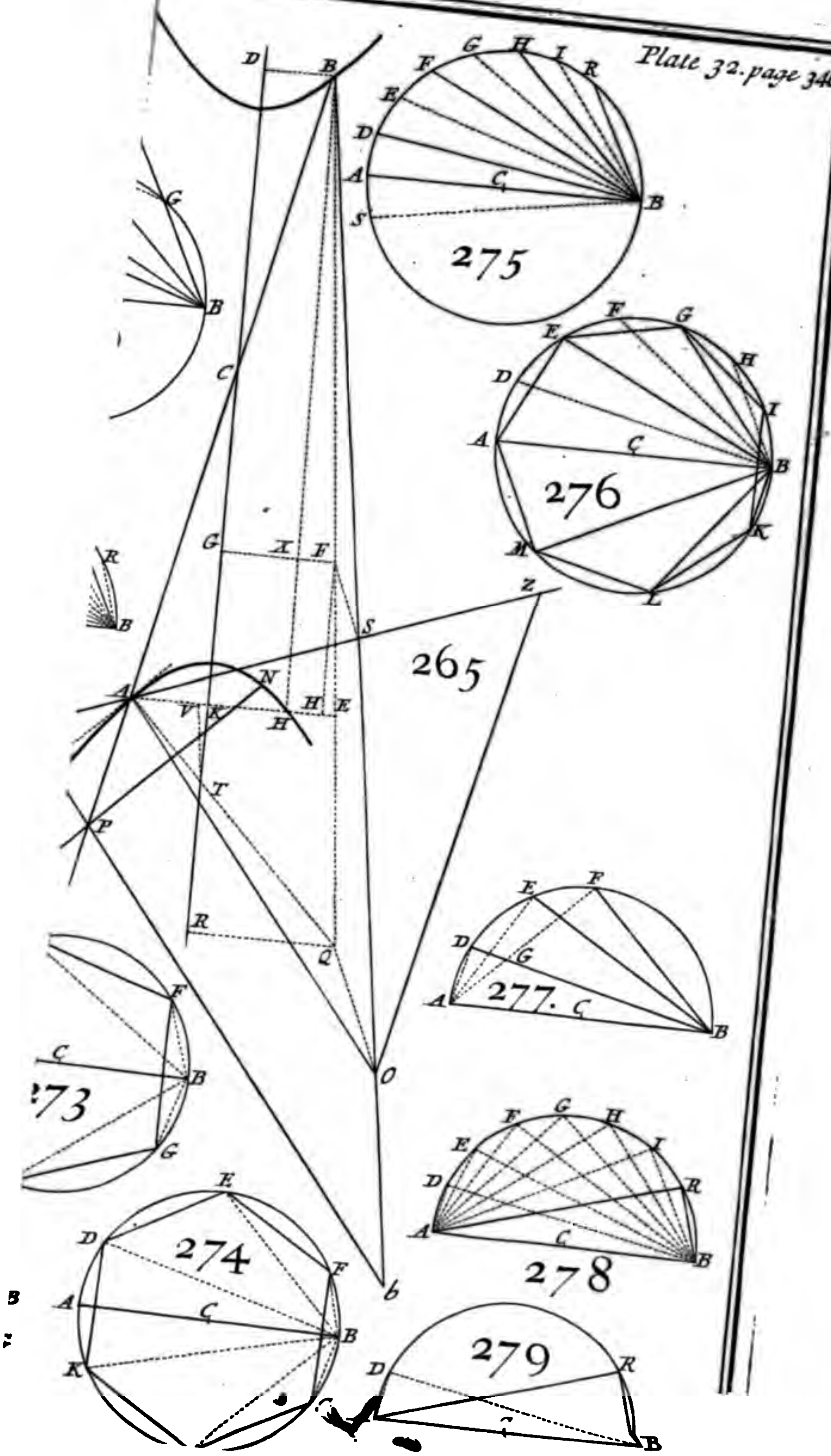
Draw the Diameter AB, and the Chords BR, AR, BD, AD, bounding the given Arc AR, and the fought Arc AD. Let the Number of equal Parts be m, the Diameter $AB = 1$, the given Chords $BR = a$, $AR = b$, and the unknown Chords $BD = x$,
*Art. 463, 475. $AD = y$. Then we fhall * have thefe two general Series $\mp a = x^m - \frac{m \times m-1}{1 \times 2} yy x^{m-2} + \frac{m \times m-1 \times m-2 \times m-3}{1 \times 2 \times 3 \times 4} y^4 x^{m-4}$, &c. $b = \frac{m}{1} y x^{m-1} - \frac{m \times m-1 \times m-2}{1 \times 2 \times 3} y^3 x^{m-3} + \frac{m \times m-1 \times m-2 \times m-3 \times m-4}{1 \times 2 \times 3 \times 4 \times 5} y^5 x^{m-5}$, &c. Wherein fubftituting the Number of equal Parts, the Arc AR is to be divided into, for m, and we fhall get two particular Equations, which being refolved will give us the fought Value of the Chord BD (x) or AD (y). For Example: If $m = 7$, then will $\mp a = x^7 - 21 yy x^5 + 35 y^4 x^3 - 7 y^6 x$, and $b = 7 y x^6 - 35 y^3 x^4 + 21 y^5 xx - y^7$, be two Equations, which muft be proceeded with as in *Art.* 462.

PROBLEM V.

FIG. 280. 478. *To find a general Equation, for the Infcription of any regular Polygon* ADEFGHK, *&c. in a Circle.*

Draw the Diameter AB, and the Chord BD, terminating the firft Side of the Polygon, and let the given Radius CA or $CB = 1$ the unknown Chord $BD = z$, and in general m be equal to half the Number of Sides of the Polygon, which I fuppofe to be odd. Then we fhall
*Art. 457, 474. have * $o = z^m - z^{m-1} - \overline{m-1} z^{m-2} + \overline{m-2} z^{m-3} + \frac{m-3}{1} \times \frac{m-2}{2} z^{m-4} - \frac{m-4}{1} \times \frac{m-3}{2} z^{m-5} - \frac{m-5}{1} \times \frac{m-4}{2} \times \frac{m-3}{3} z^{m-6} + \frac{m-6}{1} \times \frac{m-5}{2} \times \frac{m-4}{3} z^{m-7} + \frac{m-7}{1} \times \frac{m-6}{2} \times \frac{m-5}{3} \times \frac{m-4}{4} z^{m-8}$, &c. for the general Equation requir'd: of which you muft take the fame Number of Terms, as the Number $m + 1$ contains Units; becaufe what follows will be $= o$.

For Example, let 7 be the Number of Sides of the Polygon to be infcribed, then will $m = 3$, and fo $o = z^3 - zz - 2z + 1$, and the greateft Root (z) thereof fhall exprefs the Chord BD, bounding the Arc AD, whofe Chord is AD the firft Side of the Polygon. In like manner, if the Number of Sides be 11, then will $m = 5$, and fo the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the general Equation will become $o = z^5 - z^4 - 4z^3 + 3zz + 3z - 1$, whose greatest Root z is $= BD$.

PROBLEM VI.

479. *To divide a given Angle into any odd Number of equal Parts, by means of an Instrument.*

1. It is required to divide the given Angle ECF into three equal parts. Take a Rhombus $ABCD$, whose four Sides let be moveable about the four Angles, and two of them as AD, AB, let be indefinitely continued out to X and Z: fasten the Angle C of this Rhombus in the Vertex C of the given Angle ECF. And on the Sides CE, CF, mark the Points E, F, such, that CE, and CF be each equal to any one Side of the Rhombus. This being done, open or shut the Sides (AX, AZ,) of the Angle BAD, so that they pass through the Points E, F; then shall the Angle BAD be the third Part of the Angle ECF. FIG. 281, 282.

For the Triangles ABC, BCE being Isoscelles, the external Angle CBE or CEB equal to it, which is equal to the internal and opposite Angles BAC, BCA, shall be the double of the Angle BAC; and in the Triangle ECA, the external Angle ECT, which is equal to the Angle CEA *plus* the Angle BAC, shall be the triple of the Angle BAC. After the same manner we prove that the Angle FCT is the triple of the Angle DAC. Whence the given Angle ECF, is triple of the Angle BAD. *W. W. D.*

2. It is required to divide the given Angle HGK into five equal Parts. To the Angle C of the aforesaid Rhombus $ABCD$, fasten another Rhombus $CEGF$, having the Sides equal to those of the other Rhombus, and likewise moveable about their Angles. Then fix the Angle G of this latter Rhombus, in the Vertex G of the given Angle HGK, and having taken the Parts GH, GK, in the Sides of the Angle each equal to GE one Side of the Rhombus, open or shut the Anlge XAZ, moveable about the Point A, so, that its Sides AX, AZ, touch the Angle E, F, and at the same time pass through the Points H, K. I say the Angle XAZ or BAD, shall be the fifth Part of the given Angle HGK. For the Diagonal AC being drawn in the Rhombus $ABCD$, and produced at pleasure towards T; this will pass through the Point G, because the Angle ECT, FCT, being tripple of the equal Angles BAC, DAC, shall be equal to one another. But in the Triangle EGA, the external Angle HEG, which is equal to the internal and opposite ones BAC, EGA, or ECT (because CEG is an Isoscelles Triangle) shall be quadruple of the Angle BAC And therefore in the Triangle AHG, the external Angle HGT, which is equal to the internal and opposite ones BAC, FIG. 283, 284.

tion being drawn, their Interſection will give the Value of x. Or elſe the Equation $x^6 = a^5 y$ being taken as before, raiſe it to the fifth Power, and afterwards multiply it by x, and then will $x^{31} = a^{25} y^5 x = a^{30} b$, and ſo $y^5 x = a^5 b$. Whence the Locus of the Equation $x^6 = a^5 y$ being conſtructed, together with the Locus of this $y^5 x = a^5 b$, ſhall ſolve the propoſed Equation $x^{31} = a^{30} b$, ſo that you may take that of the two Loci you judge moſt ſimple. The ſame muſt be underſtood of any other Examples.

Here you muſt obſerve, that if the Dimenſion of the unknown Quantity x be not a prime Number, the propoſed Equation may be always brought lower. For Example: If $x^9 = x^8 b$, be an Equation for finding of mean Proportionals, then by extracting the Cube Root of both Sides, you will have $x^3 = \sqrt[3]{a^8 b}$. But that the Number $a^8 b$ may be a Cube, you muſt find a Line z, whoſe Cube $z^3 = aab$, or, which is the ſame thing, two mean Proportionals between a and b; for if z^3 be ſubſtituted for aab, we ſhall have $x^9 = a^6 z^3$, or $x^3 = a^2 z$, ſo that if this Equation $z^3 = aab$, be firſt ſolv'd, and then this $x^3 = aaz$, we ſhall have the Value of x being the firſt of eight mean Proportionals between a and b. In like manner, if $x^{14} = a^{13} b$ be an Equation for finding 13 mean Proportionals between a and b, by extracting the ſquare Root of both Sides we ſhall have $x^7 = \sqrt{a^{13} b}$. But that $\sqrt{a^{13} b}$ may be a Square, you muſt find a Line z, whoſe Square $zz = ab$; for if zz be ſubſtituted for ab in the propoſed Equation, then will $x^{14} = a^{12} zz$, or $x^7 = a^6 z$; whence we have only theſe two Equations to reſolve $zz = ab$, and $x^7 = a^6 z$.

You muſt obſerve farther, that theſe Equations for finding of mean Proportionals, when the Dimenſion of the unknown Quantity is a prime Number, have but one real Root; becauſe but one Line only can be the firſt of the mean Proportionals ſought.

SCHOLIUM.

481. THE aforegoing Problem may be ſolv'd by means of an Inſtrument, whoſe Conſtruction is thus. Let there be two indefinite ſtraight Lines XY, YZ, moveable about the Point Y, ſo that they may open and ſhut. Then on ſome Point B in the Side YX, fix the indefinite Perpendicular BC, which, at the Point C, wherein it cuts the other Side YZ during the opening of the Sides XY, ZY, moves the indefinite Perpendicular CD along the Side YZ; and the indefinite Perpendicular CD, at the Point D, wherein it meets the Side YX, moves the indefinite Perpendicular DE along the Side XY; and this laſt Perpendicular, at the Point E, wherein it meets the Side YZ, moves the indefinite Perpendicular EF along the Side YZ; and this laſt Perpendicular at the Point F, wherein it meets the Side YX, moves the indefinite Perpendicular FG along the Side YX; and this Perpendicular FG at the Point G, wherein it cuts the Side YZ, moves FIG. 285.

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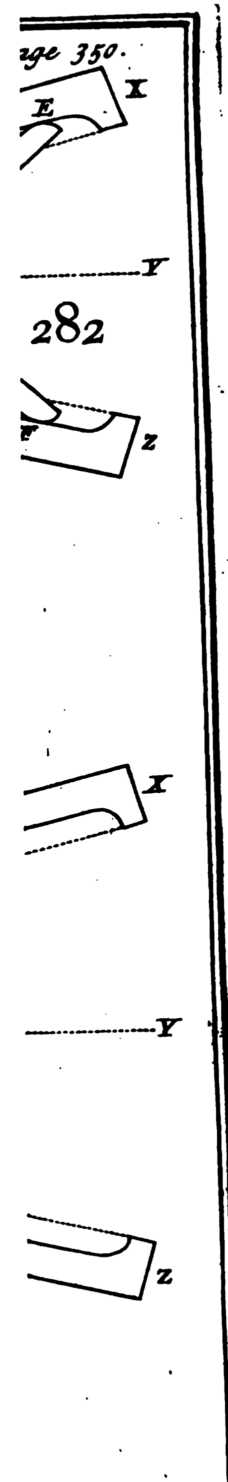
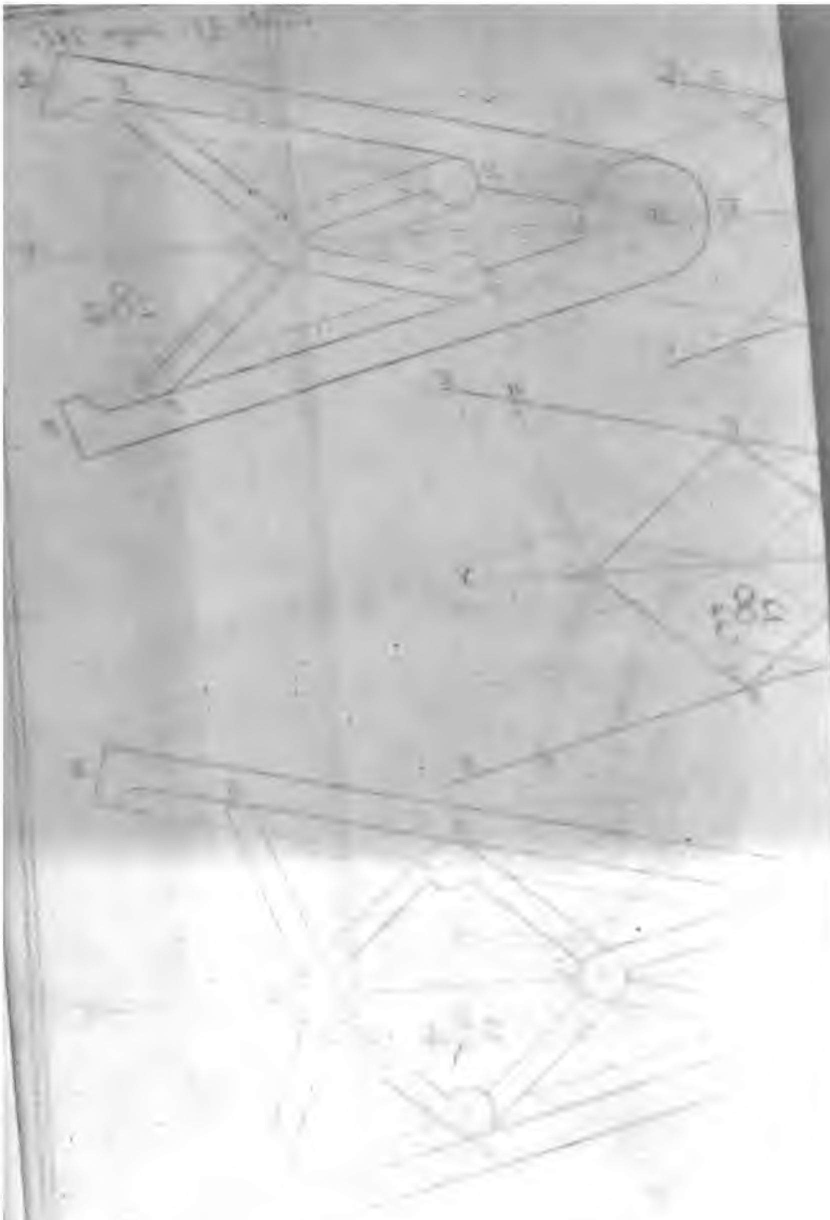
…age 350.





eighth Degree. After the ſame manner we prove, that the Curve AH is a Locus of the ſixteenth Degree, *&c.*

Now ſince according to the Example, we can find two mean Proportionals by only two Lines of the ſecond Degree; four mean Proportionals by a Locus of the ſecond Degree, and another of the third; whereas here, in the firſt Caſe, there is requir'd a Locus of the fourth Degree, being the Line AD, and a Locus of the ſecond Degree being the Circle TDE; and in the ſecond Caſe, a Locus of the eighth Degree, *viz.* the Curve AF, and a Locus of the ſecond Degree, *viz.* the Circle TFG; therefore theſe Curves AD, AF, AH, are too much compounded for ſolving the Problem, yet the Facility of their Conſtruction and Demonſtration will in ſome meaſure make up the Deficiency.

FINIS.